

CIRCLE PATTERNS, TOPOLOGICAL DEGREES AND DEFORMATION THEORY

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ABSTRACT. By using topological degree theory, this paper extends Koebe-Andreev-Thurston theorem to include obtuse intersection angles. Then it considers the characterization problem of circle patterns having the same preassigned contact graphs and dihedral angle datum. To be specific, the article develops the deformation theory by showing that the space of all such patterns could be depicted with the product of the Teichmüller spaces of its interstices. As corollaries, the density property of the set of packable surfaces and the ideal circle patterns are discussed.

Mathematics Subject Classifications (2000): 52B10, 52A15, 57Q99.

0. INTRODUCTION

The patterns of circles have become a mathematical tale after Thurston introduced them to study conformal mappings and hyperbolic geometry [45, 46]. In 1987, Rodin-Sullivan [34] proved Thurston's conjecture which implied that, under a procedure of refinement, the hexagonal circle packings would converge to the classical Riemann mapping. The subjects of circle patterns and their generalizations soon afterwards built connections to many branches, such as hyperbolic polyhedra [2, 3, 31, 32], combinatorics [39, 40], discrete and computational geometry [14, 44], minimal surfaces [5], and so forth.

Roughly speaking, a *circle pattern* \mathcal{P} on a metric surface is a connected set of circles. Particularly, if any pair of circles in \mathcal{P} are either touched or disjoint, we call \mathcal{P} a *circle packing*. The *contact graph* $G(\mathcal{P})$ of \mathcal{P} is a graph whose vertices correspond to the circles in the pattern, and an edge appears in $G(\mathcal{P})$ for every component of $\mathbb{D}_v \cap \mathbb{D}_w$, where $\mathbb{D}_v, \mathbb{D}_w$ are the closed disks bounded by a pair of circles C_v, C_w in \mathcal{P} . The *strong contact graph*, denoted by $G_{st}(\mathcal{P})$, is constructed from $G(\mathcal{P})$ by removing the edges e of $G(\mathcal{P})$, when e corresponds to the component of $\mathbb{D}_v \cap \mathbb{D}_w$ which is totally contained in the union of the closed disks bounded by the circles in \mathcal{P} aside from C_v and C_w .

Denote by V, E the set of vertices and edges in $G(\mathcal{P})$. For each $e = [u, v] \in E$, the *intersection angle* $\Theta(e)$ is defined to be the angle in $[0, \pi)$ between the clockwise tangent of C_v and the counter-clockwise tangent of C_w at a intersection point, where C_v, C_w are the circles in \mathcal{P} corresponding to vertexes v and w . For basic backgrounds on circle patterns, please refer to [15, 28, 44, 46].

Obviously, there always exist contact graph and intersection angle datum corresponding to a given circle pattern. An interesting problem is conversing this process. Assume that a preassigned graph G and a weight function $\Theta : E \mapsto [0, \pi)$

This work was partially supported by NSF of China (N0.11601141 and No.11631010).

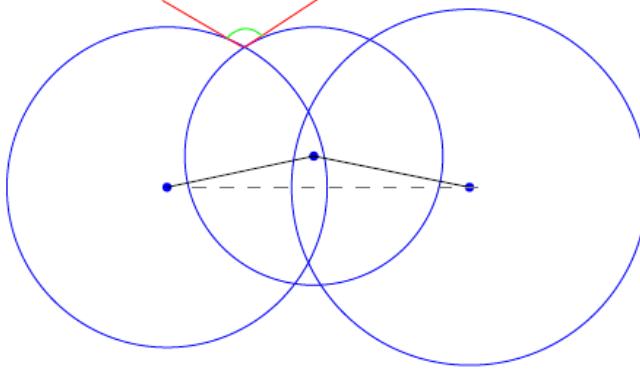


FIGURE 1.

defined in its edge set are given. In which cases are there circle patterns realizing G as contact graph and $\Theta : E \mapsto [0, \pi)$ as the intersection angle function? Moreover, suppose that the existence holds. How to characterize the solution space? The main purpose of this article is then to consider these two central questions of circle patterns. To attain the target, we shall explore into topics such as circle patterns, Riemann surfaces, topological degrees, hyperbolic geometry, variational principle and deformation theory. The method will be a combination of analysis, geometry and topology.

Notational Conventions. Throughout this paper, we use symbols $\hat{\mathbb{C}}$, \mathbb{C} , \mathbb{D} , \mathbb{H} , \mathbb{S}^2 and \mathbb{B}^3 to respectively denote the the Riemann sphere, the complex plane, the hyperbolic disk, the Poincaré half plane, the unit sphere and the hyperbolic ball. In addition, for any given set A we use the notation $|A|$ to denote the cardinality of A .

1. A BRIEF SURVEY OF RELEVANT CONSEQUENCES

A very useful form of answer to existence problem is provided by Circle Pattern theorem [15, 28, 44, 46]. In summary, it states that

Theorem 1.1 (Koebe-Andreev-Thurston). *Given a compact oriented surface S . Let G be the 1-skeleton of a triangulation of S . Suppose that $\Theta : E \mapsto [0, \pi/2]$ is a preassigned wight function satisfies*

- (i) *If a simple, null-homotopic loop in G formed by three edges e_1, e_2, e_3 , and if $\sum_{l=1}^3 \Theta(e_l) \geq \pi$, then these edges form the boundary of a triangle of G ;*
- (ii) *If a simple, null-homotopic loop in G formed by four edges e_1, e_2, e_3, e_4 , then $\sum_{l=1}^4 \Theta(e_l) < 2\pi$.*

Then there exists a constant curvature metric μ on S and a circle pattern \mathcal{P} on S with contact graph G and the intersection angles given by Θ . Moreover, the pair (μ, \mathcal{P}) is unique up to conformal equivalents.

On occasion that S is the sphere and all intersection angles are trivially equal to zero, the consequence is often called Circle Packing theorem, which could be regarded as the limiting case of Koebe's uniformization theorem saying that every finitely connected planar domain is conformal to a circle domain.

In view of the hyperbolic 3-space \mathbb{B}^3 , note that a circle in \mathbb{S}^2 is exactly the boundary of a geodesic plane of \mathbb{B}^3 . And the intersection angle of two circles is exactly equal to the dihedral angle of the corresponding geodesic planes. Owing to this observation, Theorem 1.1 is therefore closely related to Andreev's theorem [2, 3], which provides a complete characterization of compact hyperbolic polyhedra with non-obtuse dihedral angles.

For an abstract polyhedron P (or a graph G), let P^* (or G^*) denote its dual polyhedron (or graph). We call a set of edges $\gamma = \{e_1, e_2, \dots, e_k\} \subset P$ (or G) a **prismatic circuit**, if the dual edges $\{e_1^*, e_2^*, \dots, e_k^*\}$ form a simple closed curve in the dual polyhedron P^* (or G^*) and do not bound a face in P^* (or G^*). Andreev's theorem [2, 3, 35] is then stated as follows.

Theorem 1.2 (Andreev). *Let P be a trivalent polyhedron. Suppose that there exists a weight $\Theta : E \mapsto (0, \pi/2)$ assigned to its edge set such that*

- (i) *Whenever three distinct edges $\{e_l\}_{l=1}^3$ meet at a vertex, then $\sum_{l=1}^3 \Theta(e_l) > \pi$;*
- (ii) *Whenever three distinct edges $\{e_l\}_{l=1}^3$ form a prismatic 3-circuit, then $\sum_{l=1}^3 \Theta(e_l) < \pi$.*

Then there exists a hyperbolic polyhedron Q combinatorial equivalent to P with the dihedral angle function given by Θ . Moreover, this polyhedron is unique up to isometries of \mathbb{B}^3 .

The Circle Pattern theorem for higher genus surfaces was established by Thurston [46], whose proof was based on continuity method (See also [28]). And it is also Thurston that firstly find circle patterns as useful tools to build the bridge between geometry and combinatorics [46]. Furthermore, he provided the scheme to construct conformal mappings using circle packings [45].

Besides the above approaches, there are several other proofs of Theorem 1.1. For instance, Stephenson [44] provided a way using a discrete variant of Perron's method and the mathematical induction; In [15], Colin developed a variational principle method to attain the target; And Chow-Luo [12] presented a proof using the combinatoric Ricci flow which produced exponentially-convergent solutions to the circle pattern problem.

Andreev's theorem provides a way to construct hyperbolic polyhedra in terms of combinatorial type and their dihedral angles. However, the downside of this consequence is it restricts to trivalent polyhedra. So far, little results are known on constructions of other type of hyperbolic polyhedra. A relevant consequence is Rivin's theorem [31, 32], which gives an exquisite characterization of convex **ideal hyperbolic polyhedra**¹ of arbitrary combinatorial types.

Theorem 1.3 (Rivin). *Let P be a given compact convex polyhedron. Suppose that there is a weight $\Theta : E \mapsto (0, \pi)$ assigned to its edges such that*

- (i) *For each vertex v , if $\{e_l\}_{l=1}^m$ are all edges incident to v , then $\sum_{l=1}^m \Theta(e_l) = (m - 2)\pi$;*

¹ An ideal polyhedron means all its vertices are on the sphere at infinity.

(ii) For each prismatic circuit formed by $\{e_l\}_{l=1}^s$, $\sum_{l=1}^s \Theta(e_l) < (s-2)\pi$.

Then there exists a convex ideal hyperbolic polyhedra Q combinatorial equivalent to P with dihedral angle function given by Θ . Moreover, Q is unique up to hyperbolic isometries.

Recall the correspondence between hyperbolic polyhedra and circle patterns. An ideal hyperbolic polyhedra Q means a circle pattern \mathcal{P} with strong contact graph $G^*(Q)$, where $G^*(Q)$ denotes the dual graph of the 1-skeleton of Q . For a given circle pattern \mathcal{P} on a compact surface S , an *interstice* \mathbb{I} is a connected component of the complement of the union of the open disks bounded by the circles in \mathcal{P} .

Definition 1.4. A circle pattern \mathcal{P} on a compact metric surface S is called an *ideal circle pattern* if it satisfies that:

- Each interstice of \mathcal{P} consists of a single point;
- For any component \mathcal{E} of the intersection of two distinct closed disks bounded by circles in \mathcal{P} , if \mathcal{E} has non-empty intersection with the interiors of the remaining circles in \mathcal{P} , then \mathcal{E} is totally contained in the union of the closed disks bounded by the remaining circles.

In terms of *ideal circle pattern*, Rivin's theorem is then restated as follows.

Theorem 1.5 (Rivin). *Let G be the 1-skeleton of a given polyhedron. Suppose that there is a weight $\Theta : E \mapsto (0, \pi)$ assigned to its edge set such that*

- (i) *If distinct edges $\{e_l\}_{l=1}^m$ form the boundary of a face, then $\sum_{l=1}^m \Theta(e_l) = (m-2)\pi$;*
- (ii) *For each simple loop formed by $\{e_l\}_{l=1}^s$ which isn't the boundary of a face, then $\sum_{l=1}^s \Theta(e_l) < (s-2)\pi$.*

Then there exists an ideal circle pattern \mathcal{P} in \mathbb{S}^2 with strong contact graph G and dihedral angle function given by Θ . Moreover, \mathcal{P} is unique up to Möbius transformations.

Just as the ideal circle patterns are related to ideal hyperbolic polyhedra, we now introduce another class of circle patterns which are related to strictly hyperideal polyhedra. See e.g. [4, 37].

Definition 1.6. A circle pattern \mathcal{P} on a compact metric surface S is called a *strictly hyperideal circle pattern* if it satisfies that:

- Each interstice has non-empty interior;
- For any component \mathcal{E} of the intersection of two distinct closed disks bounded by circles in \mathcal{P} , if \mathcal{E} has non-empty intersection with the interiors of the remaining circles in \mathcal{P} , then \mathcal{E} is totally contained in the union of the closed disks bounded by the remaining circles;
- For each interstice \mathbb{I}_α , the geodesic lines determined by chords of the pairs of intersecting circles in $\{C_l\}_{l=1}^m$ meet at a common point in \mathbb{I}_α , where $\{C_l\}_{l=1}^m$ are the m circles adjacent to \mathbb{I}_α .

Due to the following Proposition 1.7, sometimes the third condition is replaced by

- For each interstice \mathbb{I}_α , there exists a circle C_α containing \mathbb{I}_α , which is orthogonal to every circle adjacent to \mathbb{I}_α .

Proposition 1.7. *Let $\{C_l\}_{l=1}^m$ be m ($m \geq 3$) circles in \mathbb{C} (or \mathbb{D} , \mathbb{S}^2). If there exists another circle C_0 orthogonal to all circles in $\{C_l\}_{l=1}^m$, then the geodesic lines determined by the chords of the pairs of intersecting circles in $\{C_l\}_{l=1}^m$ meet at a common point.*

As far as such kind of circle patterns concerned, in Bao-Bonahon [4], Rousset [36], and Schlenker [37], the following theorem is established.

Theorem 1.8 (Bao-Bonahon, Rousset, Schlenker). *Let S be a compact oriented surface, and G be the 1-skeleton of a cellular decomposition of S . Let $\Theta : E \mapsto (0, \pi)$ be a preassigned weigh defined on the set of edges of G such that*

- (i) *For each simple, null-homotopic loop in G formed by s edges $\{e_l\}_{l=1}^s$, then $\sum_{l=1}^s \Theta(e_l) < (s - 2)\pi$;*
- (ii) *For each open path in G formed by s edges $\{e_l\}_{l=1}^m$, which begins and end on the boundary of a face f , and is homotopic to a segment f , but is not contained in f , then $\sum_{l=1}^m \Theta(e_l) < (m - 1)\pi$.*

Then there exists a constant curvature metric μ on S and a strictly hyperideal circle pattern \mathcal{P} on S with strong contact graph G and intersection angles given by Θ . Moreover (μ, \mathcal{P}) is unique up to conformal equivalents.

Finally, we mention that Koebe's approach to Circle Packing theorem are generalized by Schramm to investigate relevant topics such as midscribable polyhedra [39], square tilings [40], and convex set packings [41, 42]. By other variational principles, Bowditch [7], Garret [16], Rivin [31, 32, 33], Leibon [25] and Springborn [5, 43], obtain a series similar results on circle patterns, hyperbolic polyhedra, Delaunay triangulations, and so on. In practice, Colins-Stephenson [13] describes a numerical algorithm for finding circle patterns, basing on ideas of Thurston. Mohar [30] shows that such method runs in time polynomial in the number of circles.

2. MAIN RESULTS

Look back on the non-obtuse conditions about the weight functions in Theorem 1.1 and Theorem 1.2. In many situations, it seems necessary to study circle patterns beyond such restrictions. In this paper, basing on topological degree theory, we shall obtain several results, where the non-obtuse conditions are relaxed.

Theorem 2.1. *Given a compact oriented surface S . Let G be the 1-skeleton of a triangulation of S . Suppose that $\Theta : E \mapsto [0, \pi)$ is a preassigned wight function satisfies*

- (i) *If a simple, null-homotopic loop in G formed by three edges e_1, e_2, e_3 , and if $\sum_{l=1}^3 \Theta(e_l) \geq \pi$, then these three edges form the boundary of a face, and $\Theta(e_1) + \Theta(e_2) \leq \pi$, $\Theta(e_2) + \Theta(e_3) \leq \pi$, $\Theta(e_3) + \Theta(e_1) \leq \pi$;*
- (ii) *For any simple, null-homotopic loop in G formed by s ($s > 3$) edges $\{e_l\}_{l=1}^s$, then $\sum_{l=1}^s \Theta(e_l) < (s - 2)\pi$.*

Then there exists a constant curvature metric μ on S and a circle pattern \mathcal{P} on S with strong contact graph G and the intersection angles given by Θ .

For any $e \in E$, denote by $I(e) = \cos \Theta(e)$. We have

Theorem 2.2. *The pair (μ, \mathcal{P}) in Theorem 2.1 is unique up to conformal equivalents if*

$$(1) \quad I(e_i) + I(e_j)I(e_k) \geq 0, \quad I(e_j) + I(e_k)I(e_i) \geq 0, \quad I(e_k) + I(e_i)I(e_j) \geq 0,$$

whenever e_i, e_j, e_k form the boundary of a face.

Remark 2.3. For those circle patterns with non-obtuse intersection angles, the conditions (i) in Theorem 2.1 and (1) in Theorem 2.2 naturally hold. Moreover, the strong contact graphs are always the same as the contact graphs. Hence the above consequences are extensions of Theorem 1.1.

Now let's consider the realization and characterization problem of circle patterns with non-triangular contact graphs. Suppose that G is the 1-skeleton of a cellular decomposition of S . Note that G could be regarded as subgraph of triangular graphs. By Theorem 1.1, it's not hard to see that there exists at least one circle packing realizing G as contact graph. However, the uniqueness doesn't hold any more. For instance, on condition that some faces of G are quadrangles, it follows from Brooks [9, 10] that up to conformal equivalent there still exist uncountable many pairs (μ, \mathcal{P}) realizing the same data. Therefore, it's of interests to ask: how to describe and characterize the problem?

The first progress was made by Brooks [9, 10]. Using continued fractions, he gave an elegant description in case that the faces of G were either triangles or quadrangles. Another approach was explored by He-Liu [20] and Huang-Liu [22]. By means of Teichmüller theory, they investigated the deformation theory of circle patterns on Riemann sphere. Moreover, it is not hard to generalize their result to circle patterns on Toruses. While the genus of the surface is $g > 1$, some difficulties seem to appear.

Let's introduce a notion as analog of the quasiconformal quadrangles. To well understand the relevant topics, some knowledge on quasiconformal mappings and Teichmüller theory is needed. For basic backgrounds, please refer to [1, 23, 24].

Given a topological polygonal domain $\mathbb{I} \subseteq \hat{\mathbb{C}}$ in the Riemann sphere, we consider all quasiconformal embeddings $h : \mathbb{I} \mapsto \hat{\mathbb{C}}$. Say two such quasiconformal embeddings $[\tau]_1, [\tau]_2 : \mathbb{I} \mapsto \hat{\mathbb{C}}$ are Teichmüller equivalent, if the composition mapping

$$[\tau]_2 \circ ([\tau]_1)^{-1} : [\tau]_1(\mathbb{I}) \mapsto [\tau]_2(\mathbb{I})$$

is homotopic to a conformal homeomorphism φ such that for each side $e_i \subset \partial\mathbb{I}$ of \mathbb{I} , φ maps $[\tau]_1(e_i)$ onto $[\tau]_2(e_i)$. Roughly speaking, there exists a **mark-preserving**² conformal mapping between their images.

Definition 2.4. The Teichmüller space of \mathbb{I} , denoted by $\mathcal{T}_{\mathbb{I}}$, is the space of all equivalence classes of quasiconformal embeddings $[\tau] : \mathbb{I} \mapsto \hat{\mathbb{C}}$.

Remark 2.5. If \mathbb{I} is m -sided ($m \geq 3$), it follows from the classical Teichmüller theory that $\mathcal{T}_{\mathbb{I}}$ is diffeomorphic to the Euclidean space \mathbb{R}^{m-3} . See e.g. [24].

Let E and F be the sets of edges and faces of G . It follows from Theorem 1.1 that there exists at least one constant curvature metric μ^0 on S and one circle packing \mathcal{P}^0 on S with contact graph G . Denote by $\{\mathbb{I}_\alpha^0\}_{\alpha=1}^{|F|}$ the set of all interstices of \mathcal{P}^0 . Evidently, each \mathbb{I}_α^0 is a polygonal domain. Let $\mathcal{T}_{G(\mathcal{P}^0)} = \prod_{\alpha=1}^{|F|} \mathcal{T}_{\mathbb{I}_\alpha^0}$. Due to Remark 2.5, we easily obtain

$$\mathcal{T}_{G(\mathcal{P}^0)} \cong \mathbb{R}^{2|E|-3|F|}.$$

²We say a homeomorphism ϕ between two m -sided polygonal domain is mark-preserving if the i -th side of one domain is mapped into the corresponding i -th side of the other.

Remark 2.6. The specific choice \mathcal{P}^0 plays a similar role to the base surface S_0 in the Teichmüller space $\mathcal{T}(S_0)$. Because the critical information of $\mathcal{T}_{G(\mathcal{P}^0)}$ depends only on the combinatoric type of G , we simply denote it as \mathcal{T}_G .

In this paper we shall prove that

Theorem 2.7. *Let G be the 1-skeleton of a cellular decomposition of a compact oriented surface S . Suppose that \mathcal{T}_G is defined as above. Assume that $\Theta : E \mapsto [0, \pi)$ is a pre-assigned weight function such that $\sum_{l=1}^s \Theta(e_l) < (s-2)\pi$, whenever $\{e_l\}_{l=1}^s$ form a simple, null-homotopic loop in G . For any*

$$[\tau] = ([\tau_1], [\tau_2], \dots, [\tau_{|F|}]) \in \mathcal{T}_G,$$

*there exists a constant curvature metric μ on S and a circle pattern \mathcal{P} on S with **strong contact graph** G and the intersection angles given by Θ , and the corresponding interstices of \mathcal{P} are endowed with the given complex structure $[\tau_i]$, $1 \leq i \leq |F|$. Moreover, the pair (μ, \mathcal{P}) is unique up to conformal equivalents.*

Turning to the limiting case, we have the following generalization of Rivin's theorem [31, 32].

Theorem 2.8. *Let G be the 1-skeleton of a cellular decomposition of a compact oriented surface S . Suppose that there is a weight $\Theta : E \mapsto (0, \pi)$ assigned to its edges set such that*

- (i) *If distinct edges $\{e_l\}_{l=1}^m$ forms the boundary of a face, then $\sum_{l=1}^m \Theta(e_l) = (m-2)\pi$;*
- (ii) *For each simple, null-homotopic loop formed by $\{e_l\}_{l=1}^s$ which isn't the boundary of a face, then $\sum_{l=1}^s \Theta(e_l) < (s-2)\pi$.*

*Then there exists a constant curvature metric μ and an ideal circle pattern \mathcal{P} on S with **strong contact graph** G and dihedral angle function given by Θ . Moreover, the pair (μ, \mathcal{P}) is unique up to conformal equivalents.*

For a compact Riemann surface, we call it packable if it supports a circle packing with triangular contact graph. As corollary of Theorem 2.7, we shall obtain that the following result which was first proved by Brooks [10].

Theorem 2.9 (Brooks). *The packable surfaces form a density set in the Teichmüller space of S . That is, every compact Riemann surface could be approximated by packable ones.*

Except the above two results, the deformation theory also plays roles in other questions. For instance, combining it with some topological techniques, in a joint work with Liu [26], the author provided an approach to an interesting problem called "how many cages midscribe an egg". Other exploration includes the stability of inscribable graphs [27].

The paper is organized as follows. In Section.3, we shall collect some preliminary results on manifolds, especially on topological degree theory. Section.4 includes several results on three-circle configurations. In Section.5, we devote to the proof of Theorem 2.1 basing on topological degree theory. By variational principle, in Section.6 Theorem 2.2 is established. Section.7 includes several prepared results on sub-patterns. Many topics, such as the deformation theory, the density property of the set of packable surfaces and ideal circle patterns are discussed in Section.8. In the last three sections, there are three appendixes complementing some details on schlicht functions and sub-patterns.

3. PRELIMINARIES

In this section, let's introduce some results on manifolds, especially on topological degree theory. Please refer to [17, 21, 29] for basic backgrounds.

Assume that M, N are two oriented smooth manifolds. Recall that a point $x \in M$ is called critical for a C^1 map $f : M \mapsto N$ if the tangent map $df : T_x \mapsto N_{f(x)}$ is not surjective. We denote by C_f the set of critical points of f . Note that $N \setminus f(C_f)$ is defined to be the set of regular values of f .

Theorem 3.1 (Regular value theorem). *Let $f : M \mapsto N$ be a C^r ($r \geq 1$) map, and let $y \in N$ be a regular value of f . Then $f^{-1}(y)$ is a closed C^r sub-manifold of M . If $y \in \text{im}(f)$, then the codimension of $f^{-1}(y)$ is equal to the dimension of N .*

Theorem 3.2 (Sard's theorem). *Let M, N be manifolds of dimensions m, n and $f : M \mapsto N$ be a C^r map. If*

$$r > \max\{0, m - n\},$$

then $f(C_f)$ has zero measure in N .

In what follows assume that M and N have equal dimensions. Let $\Lambda \subset M$ be an open subset with compact closure $\bar{\Lambda} \subset M$. We say f is transversal to y along Λ , denoted by $f \pitchfork_{\Lambda} y$, if y is a regular value of the restriction mapping $f : \Lambda \mapsto N$. Denote by $\partial\Lambda = \bar{\Lambda} \setminus \Lambda$ the boundary of Λ .

Given a continuous map $f : M \mapsto N$ and a point $y \in N$ such that $f(\partial\Lambda) \subset N \setminus \{y\}$. We will define a topological invariant $\deg(f, \Lambda, y)$, called the topological degree of f and y in Λ .

If $f \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$ such that y is a regular value, then $\Lambda \cap f^{-1}(y)$ consists of finite points. Let $x \in \Lambda \cap f^{-1}(y)$. The $\text{sgn}(f, x)$ at x is $+1$, if the tangent map $d_x f : M_x \rightarrow N_y$ preserves orientation. Otherwise, $\text{sgn}(f, x) = -1$.

Definition 3.3. Suppose that $\Lambda \cap f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$. Define that

$$\deg(f, \Lambda, y) := \sum_{j=1}^m \text{sgn}(f, x_j).$$

The following proposition implies that $\deg(f, \Lambda, y)$ has elegant property. Please refer to [17, 21, 29] for its complete proof.

Proposition 3.4. *Suppose that $f_i \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$, $f_i \pitchfork_{\Lambda} y$ and $f_i(\partial\Lambda) \subset N \setminus \{y\}$, $i = 0, 1$. If there exists a homotopy*

$$H \in C^0(I \times \bar{\Lambda}, N)$$

such that $H(0, \cdot) = f_0(\cdot)$, $H(1, \cdot) = f_1(\cdot)$, and $H(I \times \partial\Lambda) \subset N \setminus \{y\}$, then

$$\deg(f_0, \Lambda, y) = \deg(f_1, \Lambda, y).$$

The next lemma, which helps us to define the topological degrees for general maps, is a consequence of Sard's theorem.

Lemma 3.5. *For any $f \in C^0(\bar{\Lambda}, N)$ with $f(\partial\Lambda) \subset N \setminus \{y\}$, there exists $g \in C^0(\bar{\Lambda}, N) \cap C^\infty(\Lambda, N)$ and $H \in C^0(I \times \bar{\Lambda}, N)$ such that*

- (1) $g \pitchfork_{\Lambda} y$;
- (2) $H(0, \cdot) = f(\cdot)$, $H(1, \cdot) = g(\cdot)$;
- (3) $H(I \times \partial\Lambda) \subset N \setminus \{y\}$.

We are now ready to define the topological degrees of general continuous maps. Suppose that $f \in C^0(\bar{\Lambda}, N)$ with $f(\partial\Lambda) \subset N \setminus \{y\}$.

Definition 3.6. The topological degree of f and y in Λ is defined as

$$\deg(f, \Lambda, y) = \deg(g, \Lambda, y),$$

where g is given as Lemma 3.5.

It follows from Proposition 3.4 that $\deg(f, \Lambda, y)$ is well-defined. Furthermore, we have the following result on the homotopy invariance property of the topological degrees.

For $i = 0, 1$, suppose that $f_i \in C^0(\bar{\Lambda}, N)$ such that $f_i(\partial\Lambda) \subset N \setminus \{y\}$.

Theorem 3.7. *If there exists $H \in C^0(I \times \bar{\Lambda}, N)$ such that*

- (1) $H(0, \cdot) = f_0(\cdot), H(1, \cdot) = f_1(\cdot),$
- (2) $H(I \times \partial\Lambda) \subset N \setminus \{y\},$

then we have $\deg(f_0, \Lambda, y) = \deg(f_1, \Lambda, y)$.

Theorem 3.8. *Let $\gamma \subset N$ be a continuous curve such that $f(\partial\Lambda) \subset N \setminus \{\gamma\}$. Then*

$$\deg(f, \Lambda, \gamma(t)) = \deg(f, \Lambda, \gamma(0)), \quad \forall t \in [0, 1].$$

Moreover, it immediately follows from the definition that

Theorem 3.9. *If $\deg(f, \Lambda, y) \neq 0$, then we have $\Lambda \cap f^{-1}(y) \neq \emptyset$.*

4. SEVERAL RESULTS ON THREE-CIRCLE CONFIGURATIONS

The three-circle configurations are the basic building blocks of circle patterns. In this section, let's give a discussion on them.

4.1. Three lemmas on three-circle configurations. The following three results play basic roles in many proofs of Koebe-Andreev-Thurston theorem. Please refer to [15, 28, 44, 46] for detailed proofs of them.

Lemma 4.1. *For three positive numbers $r_i, r_j, r_k > 0$ and three angles*

$$\Theta_i, \Theta_j, \Theta_k \in [0, \pi/2],$$

then there exists a configuration of three circles in \mathbb{D} , unique up to hyperbolic isometries, having hyperbolic radii $\{r_i, r_j, r_k\}$ and intersection angles $\{\Theta_i, \Theta_j, \Theta_k\}$.

As in FIGURE 2, let $\vartheta_i, \vartheta_j, \vartheta_k$ be the corresponding angle of the hyperbolic triangle of centers (in hyperbolic sense) of the three circles.

Lemma 4.2. *Suppose that $\Theta_i, \Theta_j, \Theta_k \in [0, \pi/2]$ are fixed. Let $\vartheta_i, \vartheta_j, \vartheta_k$ vary with r_i, r_j, r_k . Then*

$$\frac{\partial \vartheta_i}{\partial r_i} < 0, \quad \frac{\partial \vartheta_i}{\partial r_j} > 0, \quad \frac{\partial(\vartheta_i + \vartheta_j + \vartheta_k)}{\partial r_i} < 0.$$

Lemma 4.3. *In the above case, we have*

$$(2) \quad \lim_{r_i \rightarrow \infty} \vartheta_i = 0$$

$$\begin{aligned}
(3) \quad & \lim_{(r_i, r_j, r_k) \rightarrow (0, a, b)} \vartheta_i = \pi - \Theta_i \\
& \lim_{(r_i, r_j, r_k) \rightarrow (0, 0, c)} \vartheta_i = \pi \\
& \lim_{(r_i, r_j, r_k) \rightarrow (0, 0, 0)} \vartheta_i + \vartheta_j + \vartheta_k = \pi
\end{aligned}$$

where a, b, c are fixed positive constants.

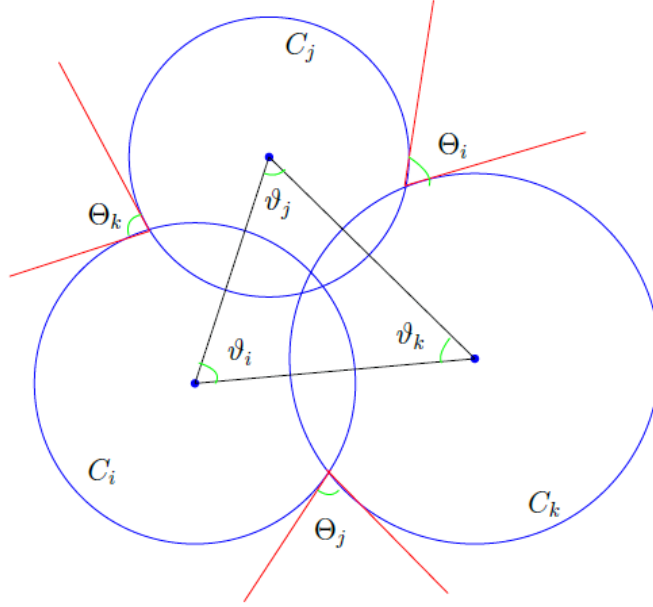


FIGURE 2.

4.2. Extended discussion. In order to generalize Theorem 1.1, firstly let's consider the extensions of the above lemmas on three-circle configurations.

Lemma 4.4. Suppose $\Theta_i, \Theta_j, \Theta_k \in [0, \pi)$ are three angles satisfies one of the following two condition:

- $\Theta_i + \Theta_j \leq \pi, \Theta_j + \Theta_k \leq \pi, \Theta_k + \Theta_i \leq \pi;$
- $\Theta_i = \Theta_j \leq \frac{\pi}{2}.$

For three positive numbers $r_i, r_j, r_k > 0$, there exists a configuration of three circles in \mathbb{D} , unique up to hyperbolic isometries, having hyperbolic radii $\{r_i, r_j, r_k\}$ and intersection angles $\{\Theta_i, \Theta_j, \Theta_k\}$.

Proof. It's equivalent to prove the triangle inequalities. In other words, we need to demonstrate that

$$(4) \quad \cosh(l_i + l_j) > \cosh l_k$$

and

$$(5) \quad \cosh(l_i - l_j) < \cosh l_k.$$

where $l_i > 0$ is defined to be

$$\cosh l_i = \cosh r_j \cosh r_k + \cos \Theta_i \sin r_j \sinh r_k.$$

And l_j, l_k are similarly defined.

Combining (4) and (5), it's equivalent to show

$$(6) \quad (\cosh l_i \cosh l_j - \cosh l_k)^2 < \sinh^2 l_i \sinh^2 l_j.$$

To simplify the notations, for $\eta = i, j, k$, denote by

$$a_\eta = \cosh r_\eta, \quad x_\eta = \sinh r_\eta, \quad I_\eta = \cos \Theta_\eta.$$

Then

$$(7) \quad \cosh l_i \cosh l_j - \cosh l_k = (a_i a_j + I_i I_j x_i x_j) x_k^2 + (I_i a_i x_j + I_j a_j x_i) a_k x_k - I_k x_i x_j.$$

Substituting (7) into (6), we then need prove that

$$(8) \quad \begin{aligned} & \left[(1 - I_i^2) a_i^2 x_j^2 + (1 - I_j^2) a_j^2 x_i^2 + 2\gamma_{kij} a_i a_j x_i x_j + (I_i^2 + I_j^2 + 2I_i I_j I_k) x_i^2 x_j^2 \right] x_k^2 \\ & + 2(\gamma_{ijk} a_j x_i + \gamma_{jki} a_i x_j) a_k x_i x_j x_k + (1 - I_k^2) x_i^2 x_j^2 > 0, \end{aligned}$$

where

$$\gamma_{ijk} = I_i + I_j I_k.$$

Suppose that the first condition holds. If all the three intersection angles are non-obtuse, then $I_\eta = \cos \Theta_\eta \geq 0$ for $\eta = i, j, k$, which immediately demonstrates (8). Otherwise, without loss of generality, suppose that

$$\Theta_i \leq \frac{\pi}{2}, \quad \Theta_j \leq \frac{\pi}{2}, \quad \Theta_k \geq \frac{\pi}{2}.$$

According to the condition $\Theta_i + \Theta_k \leq \pi$, we have

$$\begin{aligned} \gamma_{ijk} &= I_i + I_j I_k \\ &= \cos \Theta_i + \cos \Theta_j \cos \Theta_k \\ &\geq \cos \Theta_i + \cos \Theta_k \\ &= 2 \cos \frac{\Theta_i + \Theta_k}{2} \cos \frac{\Theta_i - \Theta_k}{2} \\ &\geq 0. \end{aligned}$$

Similarly,

$$\gamma_{jik} = I_j + I_i I_k \geq 0.$$

And

$$I_i^2 + I_j^2 + 2I_i I_j I_k \geq (|I_i| - |I_j|)^2 \geq 0.$$

In addition,

$$\begin{aligned} & (1 - I_j^2) a_j^2 x_i^2 + (1 - I_i^2) a_i^2 x_j^2 + 2\gamma_{kij} a_i a_j x_i x_j \\ &= \sin^2 \Theta_j a_j^2 x_i^2 + \sin^2 \Theta_i a_i^2 x_j^2 + 2(\cos \Theta_k + \cos \Theta_i \cos \Theta_j) a_i a_j x_i x_j \\ &\geq 2(\cos(\Theta_i - \Theta_j) + \cos \Theta_k) a_i a_j x_i x_j \\ &= 4 \cos \frac{\Theta_i - \Theta_j + \Theta_k}{2} \cos \frac{\Theta_i - \Theta_j - \Theta_k}{2} a_i a_j x_i x_j \\ &\geq 0. \end{aligned}$$

Hence (8) holds, which derives the triangle inequalities.

On the second condition, similar reasoning implies that (8) still holds, which thus completes the proof. \square

Lemma 4.5. *Lemma 4.3 holds in case that the three intersection angles*

$$\Theta_i, \Theta_j, \Theta_k \in [0, \pi)$$

satisfies one of the conditions in Lemma 4.1.

The proof is simple. We omit the details.

Remark 4.6. In the end, it should be pointing out that analogous results still hold in case that the background metric is Euclidean.

5. TOPOLOGICAL DEGREE AND EXISTENCE

This section is aim at the proof of Theorem 2.1. We shall mainly discuss the theorem for case that S is of genus $g > 1$. Actually the remaining cases could be shown analogously and the detailed proofs will be leaved to a following paper.

We anticipate to demonstrate the extended circle pattern theorems via similar strategies to Thurston [46] and Marden-Rodin [28]. However, since no analogous result to Lemma 4.2 is established, it's necessary to turn to a new kind of continuity method. That is, the topological degree theory.

5.1. Thurston's construction. Denote by $\{T_1, T_2, \dots, T_m\}$ the faces determined by the triangular graph G . Let $r = (r_1, r_2, \dots, r_{|V|})$ be a vector with $|V|$ positive numbers. Then r together with the given intersection angle data $\Theta : E \mapsto [0, \pi)$ satisfying the conditions (i), (ii) in Theorem 2.1 could determines a polygonal structure on S as follows.

For each face $T_\alpha \in \{T_1, T_2, \dots, T_m\}$, denote by $\{v_1, v_2, v_3\}$ its vertices. Due to Lemma 4.4, we associate the triangle determined by the centers of 3 mutually intersecting circles with the given hyperbolic radii $r_{v_1}, r_{v_2}, r_{v_3}$ and with intersection angles $\Theta([v_1, v_2]), \Theta([v_2, v_3]), \Theta([v_3, v_1])$.

Transfer the hyperbolic metrics on these hyperbolic triangles to the associated faces of $\{T_1, T_2, \dots, T_m\}$ and then paste the adjacent triangles along the common edges. Because the triangles have the same lengths on the corresponding common edges, this pasting procedure works well. In this way we obtain a metric surface which is locally hyperbolic with at most cone type singularities at the vertices. Denote this metric space by M_r . Using the metric of M_r , for $1 \leq l \leq |V|$, the discrete curvature k_l is defined to be

$$k_l := k(v_l) = 2\pi - \sum \sigma(v_l),$$

where $\sum \sigma(v_l)$ is the sum of the angle at v_l of all faces containing v_l .

Obviously, k_l are differentiable functions of Θ and r . We could write k_l as $k_l(\Theta, r)$. Moreover, if there exists r^0 such that $k_l(\Theta, r^0) = 0$ for $l = 1, 2, \dots, |V|$, then the cone singularities turn into smooth, and the surface together with the metric M_{r^0}

becomes a hyperbolic surface. The main task of this section is then to find such a hyperbolic radii label. Consider the map

$$\begin{aligned} Th_\Theta : \mathbb{R}_+^{|V|} &\mapsto \mathbb{R}^{|V|} \\ (r_1, r_2, \dots, r_{|V|}) &\mapsto (k_1(\Theta, \cdot), k_2(\Theta, \cdot), \dots, k_{|V|}(\Theta, \cdot)). \end{aligned}$$

We need to show that the point $p_0 = (0, 0, \dots, 0)$ is in the image of Th_Θ .

5.2. Topological degree. To attain the target, let's employ the topological degree theory. It's necessary to find a suitable relatively compact open set $\Lambda \subset \mathbb{R}_+^{|V|}$ and compute the degree $deg(Th_\Theta, \Lambda, p_0)$. If we could show that

$$deg(Th_\Theta, \Lambda, p_0) \neq 0,$$

it then follows from Theorem 3.9 that Theorem 2.1 holds. In order to determine $deg(Th_\Theta, \Lambda, p_0)$, we would like to use the homotopy method. Specifically, let Th_Θ continuously deform to another "good" map Th_{Θ_0} , where $deg(Th_{\Theta_0}, \Lambda, p_0)$ is easier to manipulate. In view of Theorem 3.7, the problem is then resolved through the identity

$$deg(Th_\Theta, \Lambda, p_0) = deg(Th_{\Theta_0}, \Lambda, p_0).$$

Let's construct Th_{Θ_0} . In fact, this is just done by setting $\Theta_0(e) = 0$ for every $e \in E$. Define $\Theta(t) = (1-t)\Theta_0 + t\Theta$ for $t \in [0, 1]$. Since the weight function Θ satisfies the two conditions (i), (ii) in Theorem 2.1, $\Theta(t)$ satisfies them as well. Due to Lemma 4.4, for all $t \in [0, 1]$, $Th_{\Theta(t)}$ are well-defined. Thus they form a continuous homotopy from Th_Θ to Th_{Θ_0} . Now it remains to find the suitable set Λ .

Lemma 5.1. *There exists a relatively compact open set $\Lambda \subset \mathbb{R}_+^{|V|}$ such that $Th_{\Theta(t)}(\partial\Lambda) \subset \mathbb{R}^{|V|} \setminus \{p_0\}$ for all $t \in [0, 1]$.*

Proof. Note that a point $r(t)$ in pre-image $Th_{\Theta(t)}^{-1}(p_0)$ actually corresponds to the radii of a circle pattern realizing $(G, \Theta(t))$. Denote by $(\mu(t), \mathcal{P}(t))$ the circle pattern pair realizing $(G, \Theta(t))$. Our aim is then to prove that the circles in $\mathcal{P}(t)$ wouldn't be extremely distorted as t varies in $[0, 1]$. To be specific, no circle in $\mathcal{P}(t)$ becomes infinity or degenerates to a single point.

We assume, by contradiction, that there is not such a set Λ . That means there exist a fixed index $l \in \{1, 2, \dots, |V|\}$ and sequence $\{t_n\} \subset [0, 1]$ such that, as $n \rightarrow \infty$, one of the the following two cases occurs:

- $r_l(t_n) \rightarrow +\infty$;
- $r_l(t_n) \rightarrow 0$.

In the first case, due to Lemma 4.3 and Lemma 4.5, it follows from formula (2) that $k_l(\Theta(t_n), r(t_n)) \rightarrow 2\pi$, which contradicts to the fact that $r(t_n) \in Th_{\Theta(t_n)}^{-1}(p_0)$. We thus rule out this possibility.

Turn to the second case. Let V_0 be the set of vertices v such that the corresponding circles degenerate. Denote by $Lk(V_0)$ the sub-complex spanned by the vertices in V_0 . Without loss of generality, we assume $Lk(V_0)$ is connected. Assume that $Lk(V_0)$ is of homotopy type (g_0, n_0) , where g_0, n_0 respectively denote the genus and the number of boundary components of $Lk(V_0)$.

If $g_0 > 0$, as the the circles corresponding to vertices in V_0 degenerate, there exists at least one simple geodesic curve λ in S such that the length $\ell_\lambda \rightarrow 0$. It follows from the Collar theorem (See Chap.4 in [11]) that there exists an embedding cylinder domain $Cy(\lambda)$ in S such that

$$Cy(\lambda) = \{p \in S \mid \text{dist}(p, \lambda) \leq d\},$$

where $d > 0$ satisfies

$$\sinh(\ell_\lambda/2) \sinh d = 1.$$

As $\ell_\lambda \rightarrow 0$, the cylinder $Cy(\lambda)$ becomes such thin and long that every circle in \mathcal{P}_∞ couldn't escape from it and thus degenerate to points. At last the area of the surface (S, μ_{t_n}) tends to zero, which contradicts to the Gauss-Bonnet formula.

Let $g_0 = 0$. If $n_0 > 1$, then similar reasoning to the former case also leads to contradiction.

Assume that $g_0 = 0$, $n_0 = 1$. Let $\overline{Lk}(V_0)$ be the sub-complex consisting of those faces of G having at least one vertex in V_0 . Note that the boundary $E_\partial = \{e_l\}_{l=1}^s$ of $\overline{Lk}(V_0)$ obviously forms a simple, null-homotopic curve in G which isn't the boundary of a triangle. Denote by V_∂ the set of end vertexes of all $\{e_l\}_{l=1}^s$. As those circles corresponding to V_0 degenerate, the circles corresponding to V_∂ will meet at a common point, which manifests that

$$\sum_{l=1}^s \Theta(t_n)(e_l) \rightarrow (k-2)\pi.$$

However, according to the condition (ii) in Theorem 1.1, we have

$$\sum_{l=1}^s \Theta(t_n)(e_l) \leq \sum_{l=1}^s \Theta(e_l) < (k-2)\pi.$$

This leads to contradiction. It thus completes the proof. \square

Now it's ready to determine the the degree.

Theorem 5.2. *Let Th_Θ and Λ be above. Then*

$$\deg(Th_\Theta, \Lambda, p_0) = 1 \quad \text{or} \quad \deg(Th_\Theta, \Lambda, p_0) = -1.$$

Proof. Due to Theorem 3.9, we need only compute $\deg(Th_{\Theta_0}, \Lambda, p_0)$. By Theorem 1.1, $\Lambda \cap Th_{\Theta_0}^{-1}(p_0)$ consists of a single point. Moreover, on account of Lemma 4.2, it's not hard to see that the Jacobian matrix of Th_{Θ_0} is diagonally dominant. Hence it's non-singular, which shows that

$$\deg(Th_{\Theta_0}, \Lambda, p_0) = 1 \quad \text{or} \quad \deg(Th_{\Theta_0}, \Lambda, p_0) = -1.$$

It thus completes the proof. \square

5.3. Existence. Up to now, we have accomplished the necessary prepared results. Let's proceed to prove Theorem 2.1.

Proof of Theorem 2.1. Owing to Theorem 3.9, it follows from Theorem 5.2 that there exists a hyperbolic metric μ on S and a circle pattern \mathcal{P} on S with intersection angles given by Θ . Denote by $G_{st}(\mathcal{P})$ the strong contact graph of \mathcal{P} . The remaining target is to check that $G_{st}(\mathcal{P})$ is identical with G . First, from the construction process of \mathcal{P} , it's not hard to see that there always exist edges corresponding to the edges of G . We need show that the converse also holds.

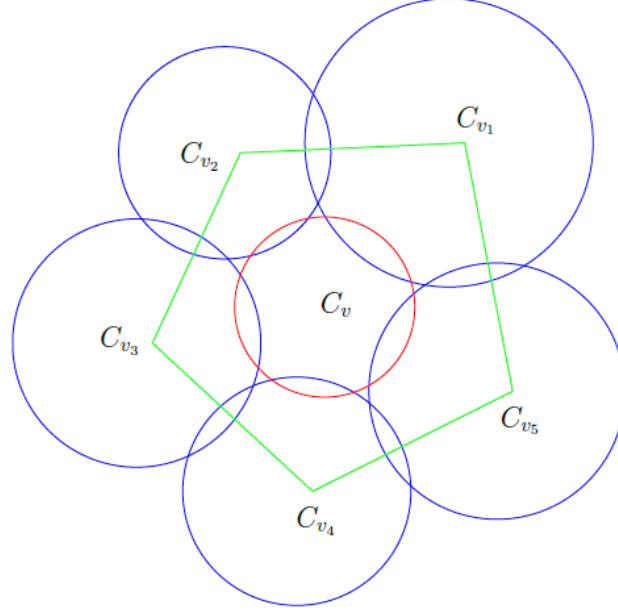


FIGURE 3.

For each $v \in V$, let $v_1, v_2, \dots, v_k \in V$ be the vertexes of G connecting to v . Connecting the centers of the circles $C_{v_1}, C_{v_2}, \dots, C_{v_k}$ via geodesic arcs, we shall obtain a hyperbolic polygon. Denote by $\overline{\text{Lk}}(v)$ the closed domain bounded by this polygon. Define that

$$\Omega_v = \overline{\text{Lk}}(v) \setminus \left(\bigcup_{i=1}^k \mathbb{D}_{v_i} \right),$$

where \mathbb{D}_{v_i} be the closed disk bounded by C_{v_i} for $i = 1, 2, \dots, k$. We call Ω_v the gap domain of v .

Now assume that $v, w \in V$ are a pair of vertexes such that the corresponding closed disks \mathbb{D}_v and \mathbb{D}_w intersect with each other. Choose a component \mathcal{E} of $\mathbb{D}_v \cap \mathbb{D}_w$. Suppose that \mathcal{E} doesn't correspond to an edge of G . If we could demonstrate that \mathcal{E} is totally contained in the union of the remaining closed disks, then the proof is completed.

Otherwise, suppose that there exists a point $x \in \mathcal{E}$ which doesn't belong to any closed disks except for \mathbb{D}_v and \mathbb{D}_w . Therefore, $x \in \Omega_v \cap \Omega_w$, where Ω_v and Ω_w are respectively the gap domains of v and w . Note that Ω_v and Ω_w has non-empty intersection if and only if there exists an edge e of G connecting v and w . This contradicts the assumption that \mathcal{E} doesn't correspond to an edge of G .

In summary, the strong contact graph $G_{st}(\mathcal{P})$ is identical with G . It thus completes the proof. \square

For sake of the correspondence between circle pattern and hyperbolic polyhedra, we have the following extended Andreev's theorem, which provides partial results on construction of hyperbolic polyhedra with obtuse dihedral angles.

Theorem 5.3. *Let P be a trivalent polyhedron. Suppose that $\Theta : E \mapsto (0, \pi)$ is a weight assigned to its edge set such that:*

- (i) *Whenever three distinct edges $\{e_l\}_{l=1}^3$ meet at a vertex, then $\sum_{l=1}^3 \Theta(e_l) > \pi$ and $\Theta(e_1) + \Theta(e_2) \leq \pi$, $\Theta(e_2) + \Theta(e_3) \leq \pi$, $\Theta(e_3) + \Theta(e_1) \leq \pi$;*
- (ii) *For any prismatic s -circuit formed by $\{e_l\}_{l=1}^s$, then $\sum_{l=1}^s \Theta(e_l) < (s - 2)\pi$.*

Then there exists a hyperbolic polyhedron Q combinatorial equivalent to P with the dihedral angle function given by Θ .

5.4. Hyperideal circle patterns. In the end, let's outline a proof of the existence of Theorem 1.8. Recall that on this occasion G is the 1-skeleton of a cellular decomposition of S . For each face of the cellular decomposition, denote by v_1, v_2, \dots, v_k all its vertexes. Now add a vertex into this face and then connect it to each v_i for $i = 1, 2, \dots, k$. Do this procedure for all faces. We thus obtain a triangulation of S . Denote by G° the 1-skeleton of this triangulation.

Let V° and E° be the sets of vertexes and edges of G° . Then

$$V^\circ = V \cup V^*, \quad E^\circ = E \cup E_{v^*v}.$$

Here V^* denotes the vertex set of the dual graph G^* of G , and the set E_{v^*v} consists of the edges v^*v , where v^* corresponds to a face and v denotes a vertex belonging to this face. Hence

$$|V^\circ| = |V| + |F|, \quad |E^\circ| = |E| + \sum_{v^* \in V^*} d(v^*) = |E| + 2|E| = 3|E|.$$

For vertexes in V and V^* , we respectively call them prismatic vertexes and newly vertexes. Similarly, the edges in E and E_{v^*v} are respectively called prismatic edges and newly edges. Without leading to ambiguity, here and hereafter we shall not distinguish a prismatic edge $e \in E$ with its corresponding edge in E_\circ .

Now assume that $\Theta : E \mapsto (0, \pi)$ is a preassigned weigh defined on the set of edges of G satisfies conditions (i), (ii) in Theorem 1.8. Let's define a new weight function $\Theta^\circ : E^\circ \mapsto (0, \pi)$ associated to the edge set of G° such that

$$\Theta^\circ(e^\circ) = \begin{cases} \Theta(e^\circ), & e^\circ \in E \\ \pi/2, & e^\circ \in E_{v^*v}. \end{cases}$$

By Proposition 1.7, we need to find circle pattern pair $(\mu^\circ, \mathcal{P}^\circ)$ realizing (G°, Θ°) . Remind the above proof of the Theorem 2.1. It's necessary to consider Thurston's construction and establish similar results to Lemma 5.1 and Theorem 5.2, which respectively depends on Lemma 4.4, Lemma 4.5 and Lemma 4.2. On condition that $\Theta : E \mapsto (0, \pi)$ satisfies (i), (ii) in Theorem 1.8, the corresponding $\Theta^\circ : E^\circ \mapsto (0, \pi)$ implies that these results do hold. In summary, we could prove the existence part of Theorem 1.8 basing on topological degree theory.

6. VARIATIONAL PRINCIPLE AND RIGIDITY

This section is devoted to the rigidity of circle patterns. Still, we merely discuss the cases that $g > 1$ and leave the remaining cases to the following paper. Our method is applying variational principle. In fact, we shall prove Theorem 2.2 via similar techniques to Guo-Luo [18] and Guo [19].

Given $\Theta_i, \Theta_j, \Theta_k \in [0, \pi)$ such that $\Theta_i + \Theta_j \leq \pi$, $\Theta_j + \Theta_k \leq \pi$, $\Theta_k + \Theta_i \leq \pi$, for any $r_i, r_j, r_k > 0$, it follows from Lemma 4.4 there exists a three-circle configuration, unique up to hyperbolic isometries, having hyperbolic radii $\{r_i, r_j, r_k\}$ and intersecting angles $\{\Theta_i, \Theta_j, \Theta_k\}$. Denote by $\vartheta_i, \vartheta_j, \vartheta_k$ the corresponding angles of triangle of centres. For $\eta = i, j, k$, let's use the substitution of variable that $u_\eta = \ln \tanh(r_\eta/2)$. In addition, recall the notational conventions in Section.4

$$a_\eta = \cosh r_\eta, \quad x_\eta = \sinh r_\eta, \quad I_\eta = \cos \Theta_\eta.$$

Note that $\vartheta_i, \vartheta_j, \vartheta_k$ are differentiable functions of $\Theta_i, \Theta_j, \Theta_k, u_i, u_j, u_k$.

Lemma 6.1. *Fixing $\Theta_i, \Theta_j, \Theta_k \in [0, \pi)$ such that*

$$\Theta_i + \Theta_j \leq \pi, \quad \Theta_j + \Theta_k \leq \pi, \quad \Theta_k + \Theta_i \leq \pi,$$

the Jacobian matrix of function $\vartheta_i, \vartheta_j, \vartheta_k$ in terms of u_i, u_j, u_k is symmetric.

Proof. Suppose $l_i, l_j, l_k > 0$ are the lengths of sides. Due to the cosine law of hyperbolic triangles,

$$\cos \vartheta_i = \frac{\cosh l_j \cosh l_k - \cosh l_i}{\sinh l_j \sinh l_k}.$$

Differentiating both sides of the equality, we obtain

$$-\sin \vartheta_i \frac{\partial \vartheta_i}{\partial l_i} = -\frac{\sinh l_i}{\sinh l_j \sinh l_k},$$

where

$$A_{ijk} = \sinh l_j \sinh l_k \sin \vartheta_i.$$

By the sine law of hyperbolic triangles, it's not hard to check that

$$A_{ijk} = A_{jki} = A_{kij} := A.$$

Hence we obtain

$$(9) \quad \frac{\partial \vartheta_i}{\partial l_i} = \frac{\sinh l_i}{A}$$

Similarly,

$$(10) \quad \frac{\partial \vartheta_i}{\partial l_j} = -\frac{\sinh l_i \cos \vartheta_k}{A}.$$

Note that

$$\cosh l_i = \cosh r_j \cosh r_k + \cos \Theta_i \sin r_j \sinh r_k = a_j a_k + I_i x_j x_k.$$

Therefore,

$$(11) \quad \frac{\partial l_i}{\partial r_j} = \frac{a_k x_j + I_i a_j x_k}{\sinh l_i}.$$

By (9), (10) and (11), we obtain

$$\begin{aligned}\frac{\partial \vartheta_i}{\partial r_j} &= \frac{\partial \vartheta_i}{\partial l_i} \frac{\partial l_i}{\partial r_j} + \frac{\partial \vartheta_i}{\partial l_k} \frac{\partial l_k}{\partial r_j} \\ &= \frac{a_k x_j + l_i a_j x_k}{A} - \frac{\sinh l_i \sinh l_k \cos \vartheta_j (a_i x_j + l_k a_j x_i)}{A \sinh^2 l_k}\end{aligned}$$

Multiplying both sides of the equation by $a_j = \sinh r_j$ and using the cosine law, we derive

$$\sinh r_j \frac{\partial \vartheta_i}{\partial r_j} = \frac{a_j \cosh l_i - a_k}{A} - \frac{(\cosh l_i \cosh l_k - \cosh l_j)(a_j \cosh l_k - a_i)}{A \sinh^2 l_k}$$

A simple computation deduces

$$\begin{aligned}& (\cosh l_i \cosh l_k - \cosh l_j)(a_j \cosh l_k - a_i) \\ &= a_j \cosh l_i \cosh^2 l_k - (a_i \cosh l_i + a_j \cosh l_j) \cosh l_k + a_i \cosh l_j \\ &= a_j \cosh l_i \sinh^2 l_k - (a_i \cosh l_i + a_j \cosh l_j) \cosh l_k + a_j \cosh l_i + a_i \cosh l_j\end{aligned}$$

It follows that

$$\sinh r_j \frac{\partial \vartheta_i}{\partial r_j} = -\frac{a_k}{A} + \frac{\Lambda_{ij} \cosh l_k - \Upsilon_{ij}}{A \sinh^2 l_k},$$

where

$$\Lambda_{ij} = a_i \cosh l_i + a_j \cosh l_j = \cosh r_i \cosh l_i + \cosh r_j \cosh l_j,$$

and

$$\Upsilon_{ij} = a_j \cosh l_i + a_i \cosh l_j = \cosh r_j \cosh l_i + \cosh r_i \cosh l_j.$$

Owing to

$$\Lambda_{ij} = \Lambda_{ji}, \quad \Upsilon_{ij} = \Upsilon_{ji},$$

it's easy to see that

$$\sinh r_j \frac{\partial \vartheta_i}{\partial r_j} = \sinh r_i \frac{\partial \vartheta_j}{\partial r_i},$$

which immediately implies that

$$\frac{\partial \vartheta_i}{\partial u_j} = \frac{\partial \vartheta_j}{\partial u_i}.$$

It thus completes the proof. \square

To continue the exploration, we need the following elementary lemma.

Lemma 6.2. Suppose $\Theta_i, \Theta_j, \Theta_k \in [0, \pi)$ are three angles satisfying

$$\cos \Theta_i + \cos \Theta_j \cos \Theta_k \geq 0, \quad \cos \Theta_j + \cos \Theta_k \cos \Theta_i \geq 0, \quad \cos \Theta_k + \cos \Theta_i \cos \Theta_j \geq 0.$$

Then

$$\Theta_i + \Theta_j \leq \pi, \quad \Theta_j + \Theta_k \leq \pi, \quad \Theta_k + \Theta_i \leq \pi.$$

Proof. First, we claim that there are at least two non-obtuse angles among $\Theta_i, \Theta_j, \Theta_k$. Otherwise, without loss of generality, assume that

$$\Theta_i, \Theta_j, \Theta_k \in (\pi/2, \pi)$$

or

$$\Theta_i \in [0, \pi/2], \quad \Theta_j, \Theta_k \in (\pi/2, \pi).$$

In the first case, we have

$$\begin{aligned} 0 &\leq \cos \Theta_i + \cos \Theta_j \cos \Theta_k < \cos \Theta_i - \cos \Theta_j. \\ \Rightarrow \cos \Theta_i &< \cos \Theta_j. \end{aligned}$$

Similarly,

$$\cos \Theta_i > \cos \Theta_j.$$

This leads to contradiction.

In the latter case, it's easy to see that

$$\cos \Theta_j + \cos \Theta_k \cos \Theta_i < 0.$$

Hence there are at least two non-obtuse angles among $\Theta_i, \Theta_j, \Theta_k$. Assume that $\Theta_i, \Theta_j \in [0, \pi/2]$. Then

$$\Theta_i + \Theta_j \leq \pi.$$

Furthermore, we have

$$0 \leq \cos \Theta_k + \cos \Theta_j \cos \Theta_i \leq \cos \Theta_k + \cos \Theta_i = 2 \cos \frac{\Theta_i + \Theta_k}{2} \cos \frac{\Theta_i - \Theta_k}{2}.$$

It's not hard to see that

$$\Theta_i + \Theta_k \leq \pi.$$

Similarly,

$$\Theta_j + \Theta_k \leq \pi.$$

It thus completes the proof. \square

Due to the above lemma, on condition that $\cos \Theta_i + \cos \Theta_j \cos \Theta_k \geq 0$, $\cos \Theta_j + \cos \Theta_k \cos \Theta_i \geq 0$, $\cos \Theta_k + \cos \Theta_i \cos \Theta_j \geq 0$, $\vartheta_i, \vartheta_j, \vartheta_k$ are well-defined. The key step of this section is to establish the following result.

Lemma 6.3. *Given $\Theta_i, \Theta_j, \Theta_k \in [0, \pi)$ such that*

$$\cos \Theta_i + \cos \Theta_j \cos \Theta_k \geq 0, \cos \Theta_j + \cos \Theta_k \cos \Theta_i \geq 0, \cos \Theta_k + \cos \Theta_i \cos \Theta_j \geq 0,$$

the Jacobian matrix of function $\vartheta_i, \vartheta_j, \vartheta_k$ in terms of u_i, u_j, u_k is negative definite.

Proof. For sake of simplicity, we use the same notational conventions as Lemma 6.1. By the cosine law of hyperbolic triangles, the following formula can be proved by direct calculation. To be specific, we have

$$\begin{pmatrix} d\vartheta_i \\ d\vartheta_j \\ d\vartheta_k \end{pmatrix} = -\frac{1}{A} \begin{pmatrix} \sinh l_i & 0 & 0 \\ 0 & \sinh l_j & 0 \\ 0 & 0 & \sinh l_k \end{pmatrix} \begin{pmatrix} -1 & \cos \vartheta_k & \cos \vartheta_j \\ \cos \vartheta_k & -1 & \cos \vartheta_i \\ \cos \vartheta_j & \cos \vartheta_i & -1 \end{pmatrix} \begin{pmatrix} dl_i \\ dl_j \\ dl_k \end{pmatrix}.$$

Differentiating the two sides of the following equality

$$\cosh l_i = \cosh r_j \cosh r_k + \cos \Theta_i \sin r_j \sinh r_k = a_j a_k + l_i x_j x_k,$$

we obtain that

$$dl_i = \frac{1}{\sinh l_i} (R_{ijk} dr_j + R_{ikj} dr_k),$$

where

$$R_{ijk} = \sinh r_j \cosh r_k + \cos \Theta_i \cosh r_j \sinh r_k = a_k x_j + l_i a_j x_k.$$

Moreover, for $\eta = i, j, k$, it follows from $u_\eta = \ln \tanh(r_\eta/2)$ that $dr_\eta = \sinh r_\eta du_\eta$.

Combining the three relations, we have

$$\begin{aligned} \begin{pmatrix} d\vartheta_i \\ d\vartheta_j \\ d\vartheta_k \end{pmatrix} &= -\frac{1}{A} \begin{pmatrix} \sinh l_i & 0 & 0 \\ 0 & \sinh l_j & 0 \\ 0 & 0 & \sinh l_k \end{pmatrix} \begin{pmatrix} -1 & \cos \vartheta_k & \cos \vartheta_j \\ \cos \vartheta_k & -1 & \cos \vartheta_i \\ \cos \vartheta_j & \cos \vartheta_i & -1 \end{pmatrix} \\ &\times \begin{pmatrix} \frac{1}{\sinh l_i} & 0 & 0 \\ 0 & \frac{1}{\sinh l_j} & 0 \\ 0 & 0 & \frac{1}{\sinh l_k} \end{pmatrix} \begin{pmatrix} 0 & R_{ijk} & R_{ikj} \\ R_{jik} & 0 & R_{jki} \\ R_{kij} & R_{kji} & 0 \end{pmatrix} \\ &\times \begin{pmatrix} \sinh r_i & 0 & 0 \\ 0 & \sinh r_j & 0 \\ 0 & 0 & \sinh r_k \end{pmatrix} \begin{pmatrix} du_i \\ du_j \\ du_k \end{pmatrix}. \end{aligned}$$

We write the above formula as

$$\begin{pmatrix} d\vartheta_i \\ d\vartheta_j \\ d\vartheta_k \end{pmatrix} = -\frac{1}{A} \mathcal{J} \begin{pmatrix} du_i \\ du_j \\ du_k \end{pmatrix},$$

where \mathcal{J} is the product of the five matrixes. Recall that we have proved \mathcal{J} is symmetric in Lemma 6.1.

To complete the proof, we need to verify that \mathcal{J} is positive definite. Firstly, let's prove that the determinant of \mathcal{J} is positive. Respectively denote the second and the fourth matrixes by S and \mathcal{R} . Then

$$\det S = -1 + \cos^2 \vartheta_i + \cos^2 \vartheta_j + \cos^2 \vartheta_k + 2 \cos \vartheta_i \cos \vartheta_j \cos \vartheta_k.$$

Since $\vartheta_i, \vartheta_j, \vartheta_k$ are the three angles of the hyperbolic triangle, it's straightforward that $\vartheta_i + \vartheta_j + \vartheta_k < \pi$. Hence

$$\begin{aligned} &\cos \vartheta_i + \cos \vartheta_j \cos \vartheta_k \\ &= \cos \vartheta_i + \cos(\vartheta_j + \vartheta_k) + \sin \vartheta_j \sin \vartheta_k \\ &= 2 \cos \frac{\vartheta_i + \vartheta_j + \vartheta_k}{2} \cos \frac{\vartheta_i - \vartheta_j - \vartheta_k}{2} + \sin \vartheta_j \sin \vartheta_k \\ &> \sin \vartheta_j \sin \vartheta_k. \end{aligned}$$

Similarly,

$$\cos \vartheta_j + \cos \vartheta_k \cos \vartheta_i > \sin \vartheta_i \sin \vartheta_k.$$

This leads to

$$\begin{aligned} &\cos^2 \vartheta_i + \cos^2 \vartheta_j + 2 \cos \vartheta_i \cos \vartheta_j \cos \vartheta_k \\ &> (\cos \vartheta_i \sin \vartheta_j + \cos \vartheta_j \sin \vartheta_i) \sin \vartheta_k \\ &= \sin(\vartheta_i + \vartheta_j) \sin \vartheta_k. \end{aligned}$$

Combining the above formulas, we derive that

$$\begin{aligned} \det S &> -\sin^2 \vartheta_k + \sin(\vartheta_i + \vartheta_j) \sin \vartheta_k \\ &= 2 \cos \frac{\vartheta_i + \vartheta_j + \vartheta_k}{2} \sin \frac{\vartheta_i + \vartheta_j - \vartheta_k}{2} \sin \vartheta_k \\ &> 0. \end{aligned}$$

It remains to consider the determinant of \mathcal{R} . A direct computation implies that

$$\det \mathcal{R} = \gamma_{ijk} Z_{ijk} + \gamma_{jki} Z_{jki} + \gamma_{kij} Z_{kij} + 2(1 + I_i I_j I_k) a_i a_j a_k x_i x_j x_k,$$

where

$$\gamma_{ijk} = I_i + I_j I_k = \cos \Theta_i + \cos \Theta_j \cos \Theta_k \geq 0,$$

$$Z_{ijk} = a_i x_i (a_k^2 x_j^2 + a_j^2 x_k^2) = \cosh r_i \sinh r_i (\sinh^2 r_j \cosh^2 r_k + \cosh^2 r_j \sinh^2 r_k) > 0.$$

It's then not hard to see

$$\det \mathcal{R} > 0,$$

which implies that

$$\det \mathcal{J} > 0.$$

Because we have proved that \mathcal{J} is symmetric, the fact $\det \mathcal{J} > 0$ implies that the sign of every eigenvalue of \mathcal{J} never changes as (r_i, r_j, r_k) varies in \mathbb{R}_+^3 . Note that \mathbb{R}_+^3 is connected. If we could prove that \mathcal{J} is positive definite at one vector, then \mathcal{J} will be positive definite in \mathbb{R}_+^3 . Write \mathcal{J} as $\mathcal{J}(\Theta, r_i, r_j, r_k)$. Pick up $(r_i, r_j, r_k) = (\rho, \rho, \rho)$. We shall show that the resulting matrix $\mathcal{J}(\Theta, \rho, \rho, \rho)$ is positive definite. For $t \in [0, 1]$, let $\Theta(t) = t\Theta$. Due to Lemma 6.2, we have

$$\Theta_i(t) + \Theta_j(t) \leq \Theta_i + \Theta_j \leq \pi.$$

Similarly,

$$\Theta_i(t) + \Theta_k(t) \leq \pi, \quad \Theta_j(t) + \Theta_k(t) \leq \pi.$$

It follows from Lemma 6.1 the matrix $J(\Theta(t), \rho, \rho, \rho)$ is well-defined. Denote by $a = \cosh \rho$, $x = \sinh \rho$. Note that

$$\det \mathcal{R}(\Theta(t), \rho, \rho, \rho) = 2(1 + \cos \Theta_i(t))(1 + \cos \Theta_j(t))(1 + \cos \Theta_k(t))a^3 x^3 > 0.$$

Hence

$$\det \mathcal{J}(\Theta(t), \rho, \rho, \rho) > 0.$$

This implies that the signs of the eigenvalues of the matrix $\mathcal{J}(\Theta(t), \rho, \rho, \rho)$ never change as t varies in $[0, 1]$.

Therefore, to prove $\mathcal{J}(\Theta, \rho, \rho, \rho)$ is positive definite, we need just check that $\mathcal{J}(\Theta(0), \rho, \rho, \rho)$ is positive definite, which could be easily derived from Lemma 4.2. In summary, it completes the proof. \square

Proof of Theorem 2.2. Let T_α be a face of the triangulation with vertexes $v_1^\alpha, v_2^\alpha, v_3^\alpha$. Consider the following 1-form

$$\omega_\alpha = \sum_{l=1}^3 \partial_{v_l^\alpha} du_{v_l^\alpha}.$$

Due to Lemma 6.1, it's not hard to see that ω_α is closed. Hence the following function

$$\Phi_\alpha = \int_{(u_i^0, u_j^0, u_k^0)}^{(x_i, x_j, x_k)} \omega_\alpha$$

is well-defined, where (u_i^0, u_j^0, u_k^0) is an arbitrary initial point.

Moreover, it follows from Lemma 6.3 that Φ_α is strictly concave, which implies that the following function

$$\Phi = \sum_\alpha \Phi_\alpha - 2\pi \left(\sum_{l=1}^{|V|} u_l \right).$$

is also strictly concave.

From Thurston's construction in Section.5, we know that the radii label of circle patterns actually correspond to the critical points of Φ . On account of the strictly concaveness of Φ , the critical point must be unique. It thus competes the proof. \square

Remark 6.4. For any given initial value $(u_1^0, u_2^0, \dots, u_{|V|}^0)$, let it evolve with the gradient flow of Φ . This will produce an exponentially convergent solution to the circle pattern metric. It is exactly the combinatorial Ricci flow method used by Chow-Luo [12].

7. SEVERAL LEMMAS ON SUB-PATTERNS

A **m -petal sub-pattern** refers to a m -sided interstice together with the circles adjacent to this interstice. Roughly speaking, it is a circle pattern with strong contact graph isomorphic to the m -polygon \mathbb{P}_m . For a m -petal sub-pattern, if the closure of its interstice consists of a single point, then we call it an **ideal m -petal sub-pattern**. In this section, we shall establish several lemmas on these configurations. Remind that the heuristic results for 3-petal case have been discussed in Sections.4. The main purpose of this part is to establish analogous results for m -petal cases.

Lemma 7.1. *Suppose that $\mathcal{P}, \tilde{\mathcal{P}} \subset \mathbb{D}$ are two m -petal sub-patterns with the same dihedral angle functions and hyperbolic radii labels. If the interstices $\mathbb{I}, \tilde{\mathbb{I}}$ of \mathcal{P} and $\tilde{\mathcal{P}}$ are endowed with the same conformal structure $[\tau]$, then $\mathcal{P}, \tilde{\mathcal{P}}$ differ up to hyperbolic motions.*

Connecting the hyperbolic centers of adjacent circles via hyperbolic geodesic arcs, we then obtain a hyperbolic polygon $\mathbb{P}_{(v_1 v_2 \dots v_m)}$. Denote by $\vartheta_1, \vartheta_2, \dots, \vartheta_m$ the corresponding angles of $\mathbb{P}_{(v_1 v_2 \dots v_m)}$ at v_1, v_2, \dots, v_m respectively. By Lemma 7.1, these angles are well-defined. Write each ϑ_i as $\vartheta_i(r_1, r_2, \dots, r_m)$, it is of interest to investigate how $\vartheta_i(r_1, r_2, \dots, r_m)$ behaves as r_1, r_2, \dots, r_m vary.

Lemma 7.2. *Let $\mathcal{P}, \tilde{\mathcal{P}} \subset \mathbb{D}$ be two m -petal sub-patterns with the same dihedral angle function. Suppose that the interstices $\mathbb{I}, \tilde{\mathbb{I}}$ of them are endowed with the same conformal structure. Denote by $\{r_l\}_{l=1}^m$ (resp. $\{\tilde{r}_l\}_{l=1}^m$) the hyperbolic radii labels of the boundary circles $\{C(v_l)\}_{l=1}^m$ (resp. $\{\tilde{C}(v_l)\}_{l=1}^m$) of \mathcal{P} (resp. $\tilde{\mathcal{P}}$). Moreover, let $\{\vartheta_l\}_{l=1}^m$ (resp. $\{\tilde{\vartheta}_l\}_{l=1}^m$) and $hy(A)$ (resp. $hy(\tilde{A})$) be the corresponding angles and area of the hyperbolic polygon of centers of the boundary circles of \mathcal{P} (resp. $\tilde{\mathcal{P}}$).*

(a) *If $r_i < \tilde{r}_i$ for some $i \in \{1, 2, \dots, m\}$, and $r_j = \tilde{r}_j$ for all $j \neq i$, then*

$$\tilde{\vartheta}_i < \vartheta_i, \quad \tilde{\vartheta}_j > \vartheta_j \ (j \neq i), \quad hy(\tilde{A}) > hy(A),$$

(b) *If $\max_{1 \leq l \leq m} \left\{ \frac{\tanh(\tilde{r}_l/2)}{\tanh(r_l/2)} \right\} = \frac{\tanh(\tilde{r}_i/2)}{\tanh(r_i/2)} > 1$, then*

$$\tilde{\vartheta}_i < \vartheta_i.$$

Remark 7.3. Using the notation of partial derivatives, the results in (i) of Lemma 7.2 could be restated as follows. For every i, j satisfying $1 \leq i \leq k, 1 \leq j \leq k$, we have³

$$\frac{\partial \vartheta_i}{\partial r_i} < 0, \quad \frac{\partial \vartheta_i}{\partial r_j} \geq 0 \ (i \neq j), \quad \frac{\partial hy(A)}{\partial r_i} > 0.$$

Lemma 7.4. $\vartheta_i \rightarrow 0$, as $r_i \rightarrow \infty$.

³The partial derivatives here are merely formal expressions of the monotonicity properties, which implies no information about differentiability.

Note that two m -petal sub-patterns are endowed with the same conformal structure if and only if there exists a mark-preserving conformal mapping between their interstices. The proofs of the above three lemmas depend on some results on schlicht functions. We defer the details to Appendix 2 and Appendix 3.

Lemma 7.5. *Given a family of m -petal sub-patterns $\{\mathcal{P}_r\} \subset \mathbb{D}$. Suppose that their dihedral angle functions and the conformal structures of interstices are fixed. Denote by $r = (r_1, r_2, \dots, r_m)$ the hyperbolic radii labels of the boundary circles $\{C_r(v_l)\}_{l=1}^m$ of \mathcal{P}_r . For some fixed $i \in \{1, 2, \dots, m\}$, if $r_i \rightarrow 0$, then the interstice \mathbb{I}_r degenerates to a single point.*

Proof. Choose one $\mathcal{P} \in \{\mathcal{P}_r\}$. Let $\{C(v_l)\}_{l=1}^m$ be the circles in \mathcal{P} . Denote by $\{\mathbb{D}(v_l)\}_{l=1}^m$ and $\{\mathbb{D}_r(v_l)\}_{l=1}^m$ the closed disks bounded by $\{C(v_l)\}_{l=1}^m$ and $\{C_r(v_l)\}_{l=1}^m$. Without loss of generality, suppose that $C_r(v_i)$ is always centered at the original point. Because $\{\mathcal{P}_r\}$ are endowed with fixed conformal structure, there always exist mark-preserving conformal mappings $\phi_r : \mathbb{I} \mapsto \mathbb{I}_r$ between the interstices of \mathcal{P} and \mathcal{P}_r . We will prove the lemma by showing that $\phi_r \rightarrow 0$ as $r_i \rightarrow 0$.

Let's consider the conformal mapping ϕ_r . We extend it to $\bar{\phi}_r : \bar{\mathbb{I}} \mapsto \bar{\mathbb{I}}_r$ by reflection with respect to $C(v_i)$ and $C_r(v_i)$. For holomorphic function family $\{\bar{\phi}_r\}$, it's not hard to see that they form a normal family. Owing to Montel's theorem, there exists a convergent subsequence $\{\bar{\phi}_{r_k}\}$ such that

$$\bar{\phi}_{r_k} \rightarrow \phi_* \quad \text{or} \quad \bar{\phi}_{r_k} \rightarrow \text{const.}$$

where ϕ_* is a holomorphic function.

We claim that the first case would never occur. Otherwise, if we choose sufficiently small disk $\mathbb{D}(z_0, \varrho)$ in $\mathbb{D}(v_i) \cap \bar{\mathbb{I}}$, then the images $\phi_r(\mathbb{D}(z_0, \varrho))$ will be contained in $\mathbb{D}_r(v_i) \cap \bar{\mathbb{I}}_r$. As $r_i \rightarrow 0$, it follows from Cauchy's integral formula that

$$|\bar{\phi}'_r(z_0)| \rightarrow 0,$$

which implies that

$$\phi'_*(z_0) = 0.$$

However, because $\bar{\phi}_r$ is univalent, ϕ_* is univalent too. Hence

$$|\phi'_*(z)| > 0, \quad \forall z \in \bar{\mathbb{I}}_0.$$

This leads to contradiction.

It follows that every subsequence of $\{\bar{\phi}_r\}$ tends to the constant function, which implies that the entire interstice \mathbb{I}_r degenerates to a single point. It thus completes the proof. \square

In the end, through direct computations, it should be pointing out that

Remark 7.6. Lemma 7.1, Lemma 7.2 and Lemma 7.4 still hold for ideal m -petal sub-patterns.

8. PROOF OF THE MAIN THEOREMS

This section is devoted to proof of Theorem 2.7, Theorem 2.8, and Theorem 2.9. As before, we assume that the genus of the S satisfies $g > 1$ and leave the remaining cases to the following paper.

8.1. Deformation theory of circle patterns. Basing on results in last section, we shall develop the deformation theory of circle patterns, which characterizes the circle patterns having the same contact graph and dihedral angles. The answer is Theorem 2.7. Now let's present the proof.

Proof of Theorem 2.7. It's divided into two parts.

Existence part. The ideal is similar to Liu-Zhou's method in [26], which is a combination of Theorem 2.1 and Rodin-Sullivan's trick [34].

For each $\mathbb{I}_i^0 \in \{\mathbb{I}_\alpha^0\}_{\alpha=1}^{|F|}$, and for any given conformal structure $[\tau_i] : \mathbb{I}_i^0 \mapsto \hat{\mathbb{C}}$, by compositing proper Möbius transformation, we may assume that the image region $[\tau_i](\mathbb{I}_i^0)$ is a bounded domain in the complex plane \mathbb{C} . Lay down the regular hexagonal circle packing in \mathbb{C} , with each circle of radius $1/n$. By using the boundary component $\partial[\tau_i](\mathbb{I}_i^0)$ like a cookie-cutter, we obtain a circle packing Q_n which consists of all circles intersecting the closed region $[\tau_i](\mathbb{I}_i^0)$. The circles in Q_n which meet the boundary $\partial[\tau_i](\mathbb{I}_i^0)$ will be called boundary circles of Q_n . Denote by $K_n(\mathbb{I}_i^0)$ the contact graph of Q_n .

Joining the contact graphs $\{K_n(\mathbb{I}_i^0)\}_{i=1}^{|F|}$ to the original graph G along the corresponding boundaries, we obtain a triangular graph G_n . Let E_n be the edge set of G_n . Define a dihedral angle function $\Theta_n : E_n \mapsto [0, \pi)$ by setting $\Theta_n(e) = \Theta(e)$ if e corresponds to an original edge in G . Otherwise, set $\Theta_n(e) = 0$. We easily check that (G_n, Θ_n) satisfies conditions (i), (ii) in Theorem 2.1. Hence there exists a hyperbolic metric μ_n on S and a circle pattern \mathcal{P}_n on S with contact graph G_n and the intersection angles given by Θ_n . Discard the circles corresponding to those vertices in every $K_n(\mathbb{I}_i^0)$. We then obtain a sequence of circle pattern pairs realizing (G, Θ) . Without leading ambiguity, we still denote them by (μ_n, \mathcal{P}_n) .

Due to Lemma 7.4, the radius of every circle in \mathcal{P}_n can't never tend to infinity, which implies that there exists an upper bound for the radii of all circles in \mathcal{P}_n as n varies. It's easy to see that there exists subsequence pairs $(\mu_{n_k}, \mathcal{P}_{n_k})$ such that $(\mu_{n_k}, \mathcal{P}_{n_k})$ converge to a pre circle pattern pair $(\mu_\infty, \mathcal{P}_\infty)$. Owing to Lemma 7.4 and Lemma 7.5, similar reasoning to Lemma 5.1 implies that no circle in $(\mu_\infty, \mathcal{P}_\infty)$ degenerates to a single point. Hence the pre circle pattern $(\mu_\infty, \mathcal{P}_\infty)$ is truly a circle pattern pair realizing (G, Θ) . Moreover, appealing to Rodin-Sullivan's demonstration in [34], it's not hard to verify that the interstices of $(\mu_\infty, \mathcal{P}_\infty)$ are endowed with the given conformal structures. Setting $(\mu, \mathcal{P}) = (\mu_\infty, \mathcal{P}_\infty)$, we then obtain the desired circle pattern pair for our theorem.

Rigidity part. We assume, by contradiction, that there're two circle pattern pairs (μ, \mathcal{P}) and $(\tilde{\mu}, \tilde{\mathcal{P}})$ satisfying the same conditions. Let r and \tilde{r} be respectively the radii labels of (μ, \mathcal{P}) and $(\tilde{\mu}, \tilde{\mathcal{P}})$. Suppose $v_0 \in V$ such that

$$\max_{v \in V} \left\{ \frac{\tanh(\tilde{r}(v)/2)}{\tanh(r(v)/2)} \right\} = \frac{\tanh(\tilde{r}(v_0)/2)}{\tanh(r(v_0)/2)} > 1.$$

Due to Lemma 7.2, it is not hard to see that

$$k(v_0) < \tilde{k}(v_0),$$

where $k(v)$ (resp. $\tilde{k}(v)$) denote the discrete curvatures associated with hyperbolic labels r (resp. \tilde{r}).

Nevertheless, according to our assumption, both (μ, \mathcal{P}) and $(\tilde{\mu}, \tilde{\mathcal{P}})$ correspond to smooth hyperbolic metric, which manifests that

$$k(v) = \tilde{k}(v) = 0, \quad \forall v \in V.$$

This leads to contradiction. Hence we always have

$$r(v) = \tilde{r}(v) = 0, \quad \forall v \in V.$$

By Lemma 7.1, the rigidity holds. \square

8.2. Ideal circle patterns. To some extent, an ideal circle pattern may be considered as the limiting case of these discussed in above part. In spirit of such an observation, let's present a proof of Theorem 2.8.

Proof of Theorem 2.8. Suppose that $\Theta : E \mapsto (0, \pi)$ is a preassigned wight function satisfies (i), (ii) in the Theorem 2.8. Let's define another weight function Θ_ε by setting $\Theta_\varepsilon(e) = \Theta(e) - \varepsilon$ for each $e \in E$, where $\varepsilon > 0$ is sufficiently close to 0 such that $\Theta_\varepsilon(e) \in (0, \pi)$. Obviously, we have

$$\sum_{l=1}^s \Theta_\varepsilon(e_l) < (s-2)\pi,$$

whenever $\{e_l\}_{l=1}^s$ forms a simple, null-homotopic loop in G .

Fix $[\tau] = ([\tau_1], [\tau_2], \dots, [\tau_{|F|}]) \in \mathcal{T}_G$. Due to Theorem 2.7, there exists circle pattern pair $(\mu_\varepsilon, \mathcal{P}_\varepsilon)$ realizing (G, Θ_ε) such that the interstices are endowed with the given complex structures $[\tau_1], [\tau_2], \dots, [\tau_{|F|}]$. To be specific, for every $1 \leq i \leq |F|$, there exists mark-preserving conformal mapping

$$[\tau_i(\varepsilon)] : \mathbb{I}_i^0 \mapsto \mathbb{I}_i(\varepsilon),$$

where \mathbb{I}_i^0 denotes the i -th interstice of the base circle pattern \mathcal{P}^0 and $\mathbb{I}_i(\varepsilon)$ denote the corresponding interstice of \mathcal{P}_ε .

It's necessary to consider how $[\tau_i(\varepsilon)]$ behaves as $\varepsilon \rightarrow 0$. By Lemma 7.4, the radius of every circle in \mathcal{P}_ε can never become infinity, which implies that there exists an upper bound for the radii of all circles in \mathcal{P}_ε as $\varepsilon \rightarrow 0$. Hence there exists subsequence pairs $(\mu_{\varepsilon_k}, \mathcal{P}_{\varepsilon_k})$ such that $(\mu_{\varepsilon_k}, \mathcal{P}_{\varepsilon_k})$ convergent to a pre circle pattern pair (μ_*, \mathcal{P}_*) . Moreover, by Montel's theorem, we assume that the conformal mapping sequence $\{[\tau_i(\varepsilon_k)]\}$ is convergent as well. As $\varepsilon_k \rightarrow 0$, then

$$[\tau_i(\varepsilon_k)] \rightarrow [\tau_i]_* \quad \text{or} \quad [\tau_i(\varepsilon_k)] \rightarrow \text{const.}$$

Here $[\tau_i]_*$ denotes a mark-preserving conformal mapping.

The first case wouldn't happen. Otherwise, suppose that the i -th interstice of \mathcal{P}_* together with the adjacent circles forms a m -petal sub-pattern. Let $\{e_l\}_{l=1}^m$ be the corresponding edges of this sub-pattern. Then

$$\lim_{k \rightarrow \infty} \sum_{l=1}^m \Theta_{\varepsilon_k}(e_l) < (m-2)\pi.$$

However, from the condition, we have

$$\lim_{k \rightarrow \infty} \sum_{l=1}^m \Theta_{\varepsilon_k}(e_l) = \sum_{l=1}^m \Theta(e_l) - \lim_{k \rightarrow \infty} m\varepsilon_k = (m-2)\pi.$$

For $i = 1, 2, \dots, |F|$, then $[\tau_i(\varepsilon_k)] \rightarrow \text{const}$, which implies that each interstice of \mathcal{P}_* consists of a single point. Thus \mathcal{P}_* is an ideal circle pattern. Let $(\mu, \mathcal{P}) = (\mu_*, \mathcal{P}_*)$, it's not hard to verify that (μ, \mathcal{P}) realizing (G, Θ) .

For the rigidity part, suppose that (μ, \mathcal{P}) and $(\tilde{\mu}, \tilde{\mathcal{P}})$ are two ideal circle pattern pairs realizing (G, Θ) . Due to Remark 7.6, similar reasoning to the proof of the rigidity part of Theorem 2.7 implies that (μ, \mathcal{P}) and $(\tilde{\mu}, \tilde{\mathcal{P}})$ are isometric. It thus completes the proof. \square

8.3. Density property of the set of packable surfaces. This is a classical result firstly established and proved by Brooks in [9, 10], and then proved by Bowers-Stephenson in [8]. In this part, we shall provide an alternative approach through Theorem 2.9.

Proof of Theorem 2.9. Denote by $\mathcal{T}(S)$ the Teichmüller space of S . For any S_μ in $\mathcal{T}(S)$, we shall show that there exists a sequence of hyperbolic metrics $\mu_n \rightarrow \mu$ such that every S_{μ_n} supports a circle packing \mathcal{P}_n with triangular contact graph.

Laying down proper circles on S_μ , it's hard to see that S_μ supports a circle packing \mathcal{P} with the contact graph $G(\mathcal{P})$ isomorphic to the 1-skeleton of a cellular decomposition of S . Denote by $\{\mathbb{I}_\alpha\}_{\alpha=1}^{|F|}$ the set of interstices of \mathcal{P} . For each $\mathbb{I}_i \in \{\mathbb{I}_\alpha\}_{\alpha=1}^{|F|}$, lifting the inclusion mapping to the universal cover \mathbb{D} , we have

$$\begin{array}{ccc} & & \mathbb{D} \\ & \nearrow [\tau_i] & \downarrow \pi \\ \mathbb{I}_i & \xrightarrow{u_i} & S_\mu \end{array}$$

The following process is almost copying the proof of Theorem 2.9. Lay down the regular hexagonal circle packing in \mathbb{C} , with every circle of radius $1/n$. By using the boundary component $\partial[\tau_i](\mathbb{I}_i)$ like a cookie-cutter, we obtain a circle packing \mathcal{Q}_n which consists of all circles intersecting the closed region $[\tau_i](\mathbb{I}_i)$. The circles in \mathcal{Q}_n which meet the boundary $\partial[\tau_i](\mathbb{I}_i)$ will be called boundary circles of \mathcal{Q}_n . Denote by $K_n(\mathbb{I}_i)$ the contact graph of \mathcal{Q}_n .

For $i = 1, 2, \dots, |F|$, let's joint the contact graph $K_n(\mathbb{I}_i)$ to the original graph G along the corresponding boundaries. We thus obtain a triangular graph G_n . From Koebe-Andreiev-Thurston theorem (Theorem 1.1), it follows that there exists a hyperbolic metric μ_n on S and a circle packing \mathcal{P}_n on S with contact graph G_n .

For sequence $\{(\mu_n, \mathcal{P}_n)\}$, due to Gauss-Bonnet formula, the radii of all circles in these patterns have an upper bound independent on n . Hence there exists a subsequence of pairs $(\mu_{n_k}, \mathcal{P}_{n_k})$ such that $(\mu_{n_k}, \mathcal{P}_{n_k})$ converges to a pre circle pattern pair $(\mu_\infty, \mathcal{P}_\infty)$. Similar reasoning to Lemma 6.1 shows that $(\mu_\infty, \mathcal{P}_\infty)$ is a non-degenerating circle packing with contact graph G . Moreover, appealing to Rodin-Sullivan's demonstration in [34], it's not hard to verify that the interstices of $(\mu_\infty, \mathcal{P}_\infty)$ are endowed with the same conformal structures as these of (μ, \mathcal{P}) .

It follows from the rigidity part of Theorem 2.7 that S_{μ_∞} and S_μ are isometric. It thus completes the proof. \square

9. APPENDIX 1: OVERVIEW ON SCHLICHT FUNCTIONS

In this appendix, some preliminary results on schlicht functions will be introduced.

Suppose that f is a holomorphic function in \mathbb{D} such that $|f(z)| < 1$. Define that

$$M(f; z) := \frac{|f'(z)|(1 - |z|^2)}{1 - |f(z)|^2}, \quad H(f; z) := \log M(f; z).$$

Here are several properties of them.

- For every β in the isometry group $Aut(\mathbb{D})$ of the hyperbolic disk \mathbb{D} ,

$$(12) \quad M(\beta; z) \equiv 1;$$

- For every $\beta, \alpha \in Aut(\mathbb{D})$,

$$(13) \quad M(\beta \circ f \circ \alpha; z) = M(f; \alpha(z));$$

- Suppose that f is invertible. Denote $w = f(z)$. Then

$$(14) \quad M(f; z)M(f^{-1}; w) \equiv 1.$$

In addition, the following two formulas will be useful.

$$(15) \quad \Delta \log \frac{1}{1 - |z|^2} = \left(\frac{2}{1 - |z|^2} \right)^2.$$

$$(16) \quad \Delta \log \frac{|f'(z)|}{1 - |f(z)|^2} = \left(\frac{2|f'(z)|}{1 - |f(z)|^2} \right)^2.$$

The next result comes from [38]. It plays an important role in He-Liu's paper [20].

Lemma 9.1. *Suppose Ω (resp. $\tilde{\Omega}$) $\subset \mathbb{C}$ is a domain bounded by N close analytic curves C_v ($v = 1, \dots, N$) (resp. Γ_v ($v = 1, \dots, N$)), now $w = f(z)$ is a univalent conformal mapping from Ω to $\tilde{\Omega}$ and carries the curve system C_v to corresponding curve system curves Γ_v . We assume $f(z)$ is analytic in the closed domain $\tilde{\Omega} = D \cup \partial\Omega$, where $\partial\Omega = \bigcup C_v$. Suppose that $z = z(s)$ and $w = w(\sigma)$ are parametric representation in terms of their arc length, then we have the following formula:*

$$\frac{\partial}{\partial n} \log |f'(z)| = k(s) - \tilde{k}(\sigma)|f'(z)|$$

where the operator $\partial/\partial n$ denotes differential with respect to the exterior normal on the boundary curves $C = \bigcup C_v$, and $k(s), \tilde{k}(\sigma)$ are curvatures of curve $C = \bigcup C_v$ and $\Gamma = \bigcup \Gamma_v$ in corresponding parameter point.

10. APPENDIX 2: RIGIDITY AND UNIQUENESS

In this part we shall present the proof of Lemma 7.1. Recall that a homeomorphism ϕ between two m -sided interstice $\mathbb{I}, \tilde{\mathbb{I}}$ is called mark-preserving if the i -th side of \mathbb{I} is mapped into the i -th side, for every $1 \leq i \leq m$.

Proof of Lemma 7.1. Because $\mathbb{I}, \tilde{\mathbb{I}}$ are endowed with the same conformal structure, there exists a mark-preserving conformal mapping $\phi : \mathbb{I} \mapsto \tilde{\mathbb{I}}$ between them. Let's consider the function $H(\phi; z)$. Suppose it attains its minimum at a point z_0 . There are four cases to distinguish.

(I). If z_0 locates in the interior of \mathbb{I} , then we must have

$$\Delta H(\phi; z_0) \geq 0.$$

Note that

$$H(\phi; z_0) = \log \frac{|\phi'(z_0)|(1 - |z_0|^2)}{1 - |\phi(z_0)|^2} = \log \frac{|\phi'(z_0)|}{1 - |\phi(z_0)|^2} - \log \frac{1}{1 - |z_0|^2}.$$

From (15) and (16), we derive that

$$\left(\frac{2|\phi'(z_0)|}{1 - |\phi(z_0)|^2} \right)^2 - \left(\frac{2}{1 - |z_0|^2} \right)^2 \geq 0,$$

which implies

$$M(\phi; z_0) \geq 1.$$

Since z_0 is the minimal point of $H(\phi; z)$,

$$H(\phi; z) \geq H(\phi; z_0) = \log M(\phi; z_0) \geq 0.$$

Equivalently,

$$M(\phi; z) \geq 1.$$

(II). If the minimal point z_0 locates in the boundary \mathbb{I} , we assume $z_0 \in C(v_j)$ for some $1 \leq j \leq m$. By compositing proper hyperbolic isometries $\beta, \alpha \in \text{Aut}(\mathbb{D})$, we may assume $C(v_j)$ and $\tilde{C}(v_j)$ are the same circle centering at the original point of Euclidean radius $r_j = \tanh(r_j/2) = \tanh(\tilde{r}_j/2)$. Owing to (13), without leading ambiguity, we would still use notation $M(\phi; z)$ instead of $M(\beta \circ \phi \circ \alpha; z)$. On account that z_0 and $\phi(z_0)$ respectively locate in $C(v_j)$ and $\tilde{C}(v_j)$, we have

$$(17) \quad |z_0| = |\phi(z_0)| = r_j.$$

Since z_0 is the minimal point, then we have

$$(18) \quad \frac{\partial}{\partial n} H(\phi; z_0) \leq 0.$$

It follows from Lemma 9.1 that

$$(19) \quad \frac{\partial}{\partial n} \log |\phi'(z_0)| = \frac{1}{r_j} - \frac{1}{r_j} |\phi'(z_0)|.$$

A direct calculation implies that

$$(20) \quad \frac{\partial}{\partial n} \log(1 - |z_0|^2) = \frac{2|z_0|}{1 - |z_0|^2},$$

$$(21) \quad \left| \frac{\partial}{\partial n} \log(1 - |\phi(z_0)|^2) \right| \leq \frac{2|\phi(z_0)||\phi'(z_0)|}{1 - |\phi(z_0)|^2}.$$

Note that

$$\frac{\partial}{\partial n} H(\phi; z_0) = \frac{\partial}{\partial n} \log |\phi'(z_0)| + \frac{\partial}{\partial n} \log(1 - |z_0|^2) - \frac{\partial}{\partial n} \log(1 - |\phi(z_0)|^2).$$

From (17), (18), (19), (20), (21), we deduce that

$$|\phi'(z_0)| \geq 1.$$

That means

$$M(\phi; z_0) \geq 1.$$

Because z_0 is the point where $M(\phi; z_0)$ attain the minimality, for all $z \in \mathbb{I}$, we have

$$M(\phi; z) \geq 1.$$

- (III). If z_0 is an intersecting point of two adjacent circles $C(v_i), C(v_{i+1})$ with the dihedral angle $\Theta_i \neq 0$. First, to make sure that the exterior normal derivative at z_0 is well-defined, we would extend the conformal map ϕ by reflection principle. To be specific, we reflect with respect to $C(v_i)$ and $C(v_{i+1})$. Thus we extend ϕ to a new conformal mapping $\bar{\phi}$ with its domain have a pair of newly circles tangent at z_0 as boundary components. Let us continue to reflect with respect to these two newly circles, and repeat this procedure until the exterior normal vector \mathbf{n}_1 and \mathbf{n}_2 of $C(v_i)$ and $C(v_{i+1})$ at z_0 belong to the domain of our extended conformal mapping $\bar{\phi}$. We would have

$$\frac{\partial}{\partial \vec{v}} H(\phi; z_0) \leq 0.$$

where \vec{v} denotes the vector which divide the dihedral angle Θ_i equally.

Furthermore, by a simple computation, we know that

$$\frac{\partial}{\partial \vec{v}} H(\phi; z_0) = \frac{1}{2 \sin(\Theta_i/2)} \left(\frac{\partial}{\partial n_1} H(\phi; z_0) + \frac{\partial}{\partial n_2} H(\phi; z_0) \right).$$

That means

$$\frac{\partial}{\partial n_1} H(\phi; z_0) \leq 0.$$

Or

$$\frac{\partial}{\partial n_2} H(\phi; z_0) \leq 0.$$

This case is then included into the former case in (II).

- (IV). If z_0 is an intersecting point of two adjacent circles $C(v_i), C(v_{i+1})$ with the dihedral angle $\Theta_i = 0$. Similarly, by compositing proper hyperbolic isometries, we may assume $z_0 = \phi(z_0) = 0$, i.e $C(v_i), C(v_{i+1})$ (resp. $\tilde{C}(v_i), \tilde{C}(v_{i+1})$) tangent at the original point. Then we would derive that:

$$M(\phi, z_0) = |\phi'(0)|.$$

By definition,

$$\phi'(0) = \lim_{z \rightarrow 0} \phi(z)/z = \lim_{z \rightarrow 0} \frac{1/z}{1/\phi(z)} = \lim_{w \rightarrow \infty} w/g(w).$$

where w and $g(w)$ are images of z and $\phi(z)$ after the map $w = 1/z$.

Actually $C(v_i), C(v_{i+1})$ will become two parallel horizontal lines with width (in Euclidean sense)

$$\frac{1}{\tanh(r_i/2)} + \frac{1}{\tanh(r_{i+1}/2)}$$

after the map $w = 1/z$, where r_i, r_{i+1} denote the hyperbolic radius of $C(v_i), C(v_{i+1})$.

Respectively, $\tilde{C}(v_i), \tilde{C}(v_{i+1})$ become parallel lines with width

$$\frac{1}{\tanh(\tilde{r}_i/2)} + \frac{1}{\tanh(\tilde{r}_{i+1}/2)}.$$

According to conditions $r_i = \tilde{r}_i$ and $r_{i+1} = \tilde{r}_{i+1}$, an easy computation implies that:

$$\lim_{w \rightarrow \infty} w/g(w) = \left(\frac{1}{\tanh(r_i/2)} + \frac{1}{\tanh(r_{i+1}/2)} \right) / \left(\frac{1}{\tanh(\tilde{r}_i/2)} + \frac{1}{\tanh(\tilde{r}_{i+1}/2)} \right) = 1.$$

Thus we know

$$M(\phi; z) \geq 1.$$

To summarise, from the above 4 cases, we always have

$$M(\phi; z) \geq 1, \forall z \in \mathbb{I}.$$

Similarly, we obtain that

$$M(\phi_{-1}; w) \geq 1, \forall w \in \tilde{\mathbb{I}}.$$

From (14), then

$$M(\phi; z) \leq 1, \forall z \in \mathbb{I}.$$

At last we derive that

$$M(\phi; z) = 1, \forall z \in \mathbb{I}.$$

Now if we define the hyperbolic metric $ds = \frac{2|dz|}{1-|z|^2}$ on \mathbb{I} , and similarly $d\tilde{s} = \frac{2|dw|}{1-|w|^2}$ on $\tilde{\mathbb{I}}$, then the fact $M(\phi; z) = 1$ implies that $\phi^* d\tilde{s} = ds$. It thus completes the proof. \square

11. APPENDIX 3: MONOTONICITY

The main purpose of this appendix is to show Lemma 7.2, which reveals several monotonicity properties of m -petal sub-patterns as the radii of the boundary circles vary.

Proof of Lemma 7.2. Similarly, there exists a mark-preserving conformal mapping $\phi : \mathbb{I} \mapsto \tilde{\mathbb{I}}$ between the interstices. As in FIGURE 4, we divide ϑ_i into three sectors $\vartheta_{i0}, \vartheta_{i1}, \vartheta_{i2}$. By definition, $\vartheta_i = \vartheta_{i0} + \vartheta_{i1} + \vartheta_{i2}$. Similarly, $\tilde{\vartheta}_i = \tilde{\vartheta}_{i0} + \tilde{\vartheta}_{i1} + \tilde{\vartheta}_{i2}$. According to conditions $r_i < \tilde{r}_i$ and $r_j = \tilde{r}_j$ ($j \neq i$), it is easy to see

$$(22) \quad \tilde{\vartheta}_{i1} < \vartheta_{i1}, \quad \tilde{\vartheta}_{i2} < \vartheta_{i2}.$$

Let's consider $\tilde{\vartheta}_{i0}$ (resp. ϑ_{i0}). Using similar method to Appendix 2, it is not hard to deduce that

$$(23) \quad 1 \leq M(\phi; z) < \frac{\tanh(\tilde{r}_i/2)}{\tanh(r_i/2)}.$$

Suppose $M(\phi; z)$ attains its minimum at some point, the left inequality is evident. Similarly, assume that $M(\phi^{-1}; w)$ attains its minimum at some point w_0 . For case (I), we will have

$$M(\phi^{-1}; w) \geq 1.$$

For case (II) or (III), it is

$$M(\phi^{-1}; w) \geq \frac{\tanh r_i}{\tanh \tilde{r}_i}.$$

For case (IV), we would deduce

$$M(\phi^{-1}; w) \geq \left(\frac{1}{\tanh(\tilde{r}_i/2)} + \frac{1}{\tanh(\tilde{r}_{i+1}/2)} \right) / \left(\frac{1}{\tanh(r_i/2)} + \frac{1}{\tanh(r_{i+1}/2)} \right).$$

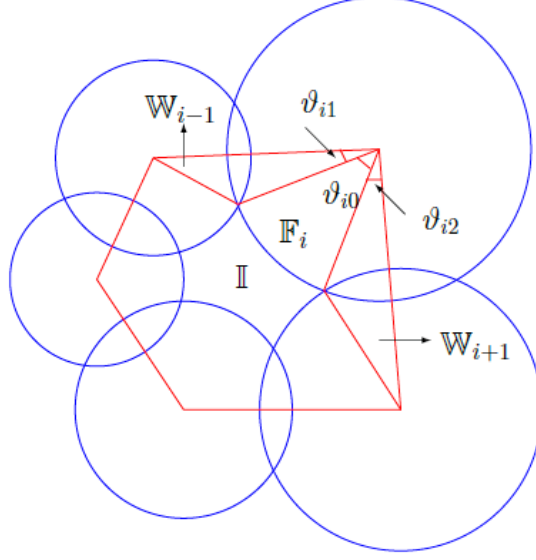


FIGURE 4.

These four cases all imply that

$$M(\phi^{-1}; w) > \frac{\tanh(\tilde{r}_i/2)}{\tanh(r_i/2)}.$$

Due to (14), we conclude our inequality (23). Note that $\phi^* d\tilde{s} = M(\phi; z)ds$. Integrating along the circular curve γ_i (resp. $\tilde{\gamma}_i$) of the boundary component of \mathbb{I} (resp. $\tilde{\mathbb{I}}$) corresponding to v_i (resp. \tilde{v}_i) and comparing both sides, we obtain

$$(24) \quad \int_{\tilde{\gamma}_i} d\tilde{s} < \frac{\tanh(\tilde{r}_i/2)}{\tanh(r_i/2)} \int_{\gamma_i} ds.$$

Moreover, an easy calculation reveals

$$\sinh r_i \vartheta_{i0} = \int_{\gamma_i} ds,$$

and

$$\sinh \tilde{r}_i \tilde{\vartheta}_{i0} = \int_{\tilde{\gamma}_i} d\tilde{s}.$$

At last, we derive

$$(25) \quad \vartheta_{i0} > \frac{\cosh^2(\tilde{r}_i/2)}{\cosh^2(r_i/2)} \tilde{\vartheta}_{i0} > \tilde{\vartheta}_{i0}.$$

Combining the inequalities (22) and (25), we show that

$$(26) \quad \vartheta_i > \tilde{\vartheta}_i.$$

Similarly, we obtain the other inequality

$$(27) \quad \vartheta_j < \tilde{\vartheta}_j \quad (j \neq i)$$

The last inequality is about area, we also divide it into sum of sectors.

$$\text{hy}(A) = A_{\mathbb{I}} + \sum_j A_{\mathbb{W}_j} + \sum_{j \neq i} A_{\mathbb{F}_j} + A_{\mathbb{F}_i}$$

where $A_{\mathbb{I}}$ denotes the area of the interstice \mathbb{I} , $\sum_j A_{\mathbb{W}_j}$ is the sum over areas of all wedge-sector, $\sum_{j \neq i} A_{\mathbb{F}_j}$ means the sum over areas of all fan-shaped sector except the i -th one, and $A_{\mathbb{F}_i}$ represents the area of the corresponding i -th fan-shaped sector. Remind that all areas here are hyperbolic sense. Similarly, we have

$$\text{hy}(\tilde{A}) = \tilde{A}_{\mathbb{I}} + \sum_j \tilde{A}_{\tilde{\mathbb{W}}_j} + \sum_{j \neq i} \tilde{A}_{\tilde{\mathbb{F}}_j} + \tilde{A}_{\tilde{\mathbb{F}}_i}$$

First, since we have proved $M(z) \geq 1$ for all $z \in I$, we directly derive that

$$\tilde{A}_{\mathbb{I}} \geq A_{\mathbb{I}}$$

By a comparison of lengths of sides in every wedge triangle, it's easy to see

$$\sum_j \tilde{A}_{\tilde{\mathbb{W}}_j} > \sum_j A_{\mathbb{W}_j}.$$

In addition, from the proved fact in (27), we know that

$$\sum_{j \neq i} \tilde{A}_{\tilde{\mathbb{F}}_j} \geq \sum_{j \neq i} A_{\mathbb{F}_j}$$

It remains to consider the last terms $A_{\mathbb{F}_i}$ and $\tilde{A}_{\tilde{\mathbb{F}}_i}$. Recall the process to prove the first inequality in (26). A key step is to derive (25) from the (23). In fact, we only use half truths of (23), i.e $M(z) < \frac{\tanh(\tilde{r}_i/2)}{\tanh(r_i/2)}$. Fortunately, from the other half truths $M(z) \geq 1$ of (23), we can derive that

$$(28) \quad \tilde{\vartheta}_{i0} \geq \vartheta_{i0} \frac{\sinh r_i}{\sinh \tilde{r}_i}.$$

By an easy computation, we have

$$A_{\mathbb{F}_i} = 2 \sinh^2 \frac{r_i}{2} \vartheta_{i0}.$$

Similarly,

$$\tilde{A}_{\tilde{\mathbb{F}}_i} = 2 \sinh^2 \frac{\tilde{r}_i}{2} \tilde{\vartheta}_{i0}.$$

Then

$$\frac{\tilde{A}_{\tilde{\mathbb{F}}_i}}{A_{\mathbb{F}_i}} = \frac{2 \sinh^2(\tilde{r}_i/2) \tilde{\vartheta}_{i0}}{2 \sinh^2(r_i/2) \vartheta_{i0}} \geq \frac{\tanh(\tilde{r}_i/2)}{\tanh(r_i/2)}.$$

Thus we deduce

$$\tilde{A}_{\tilde{\mathbb{F}}_i} > A_{\mathbb{F}_i}.$$

That means

$$\text{hy}(\tilde{A}) > \text{hy}(A).$$

Hence we complete our proofs for all three inequalities. Moreover, by analogous procedure, we could prove the inequality in (b). \square

In the end of this section, let's prove Lemma 7.4.

Proof of Lemma 7.4. Follow the above method, we write it as $\vartheta_i = \vartheta_{i0} + \vartheta_{i1} + \vartheta_{i2}$. Clearly, $\vartheta_{i1} \rightarrow 0$ and $\vartheta_{i2} \rightarrow 0$ as $r_i \rightarrow +\infty$. Moreover, it follows from (25) and (28) that $\vartheta_{i0} \rightarrow 0$ as $r_i \rightarrow +\infty$. It thus completes the proof. \square

12. ACKNOWLEDGEMENT

The author would like to thank Jinsong Liu and Yueping Jiang for encouragement and helpful comments. He also thanks to Liu Liu for the help of FIGURE 4.

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